Folded Variance Estimators for Stationary Time Series

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Folded Variance Estimators for Stationary Time Series

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To my impatient mother,

who never understood what was taking me so long.

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SUMMARY

This thesis is concerned with simulation output analysis. In particular, we are interested in estimating the *variance parameter* of a steady-state output process. The estimation of the variance parameter has immediate applications in problems involving (i) the precision of the sample mean as a point estimator for the steady-state mean μ_X, and (ii) confidence intervals for μ_X. The thesis focuses on new variance estimators arising from Schruben's method of standardized time series (STS). The main idea behind STS is to let such series converge to *Brownian bridge* processes; then their properties are used to derive estimators for the variance parameter. Following an idea from Shorack and Wellner, we study different levels of *folded* Brownian bridges. A folded Brownian bridge is obtained from the standard Brownian bridge process by "folding" it down the middle and then "stretching" it so that it spans the interval [0,1]. We formulate the folded STS, and deduce a simplified expression for it. Similarly, we define the weighted area under the folded Brownian bridge, and we obtain its asymptotic properties and distribution. We study the square of the weighted area under the folded STS (known as the *folded area estimator*) and the weighted area under the square of the folded STS (known as the *folded Cramér–von Mises, or CvM, estimator*) as estimators of the variance parameter of a stationary time series. In order to obtain results on the bias of the estimators, we provide a complete finite-sample analysis based on the mean-square error of the given estimators. Weights yielding first-order unbiased estimators are found in the area and CvM cases. Finally, we perform *Monte Carlo* simulations to test the efficacy of the new estimators on a test bed of stationary stochastic processes, including the first-order moving average and autoregressive processes and the waiting time process in a single-server Markovian queuing system.

CHAPTER I

INTRODUCTION

An objective of many computer simulation studies is the estimation of the steady-state mean $\mu_X \equiv \mathrm{E}(X_i)$ of a stationary discrete-time stochastic process $\{X_i,\ i \geq 1\}$, with the final goal of optimizing the performance of the system. In general, the sample mean $\overline{X}(n) \equiv \sum_{i=1}^{n} X_i/n$ of n observations has been widely used as an estimator for the mean μ_X. This is due to its favorable properties such as unbiasedness and consistency. The first property is due to the equality $\mathrm{E}(\overline{X}(n)) = \mu_X$. On the other hand, Khinchin proved (see Feller [14] or Lehmann [32]) that the sample mean $\overline{X}(n)$ is a consistent estimator for μ_X, as long as the existence of the expectation is guaranteed. In fact, not even the assumption of finite variance is needed for the consistency of $\overline{X}(n)$. However, since we often require a measure of the precision of $\overline{X}(n)$ as an estimator of μ_X as well, the overall estimation problem becomes a much more difficult one—especially in the context of correlated output as is usually the case in a simulation run. See Alexopoulos and Seila [3] for a variety of practical examples supporting the previous statement.

Let $R_i \equiv \mathrm{Cov}(X_1, X_{1+i})$, $i \geq 0$, be the autocovariance function of $\{X_i\}$ and let $\rho_i \equiv \mathrm{Corr}(X_1, X_{1+i}) = R_i/\sigma_X^2$, $i \geq 0$, be the autocorrelation function, where $\sigma_X^2 \equiv \mathrm{E}[(X_1 - \mu_X)^2]$. Using some algebra (see Anderson [5]), one has

$$\mathrm{Var}(\overline{X}(n)) = \frac{\sigma_X^2}{n}\left[1 + 2\sum_{i=1}^{n-1}\left(1 - \frac{i}{n}\right)\rho_i\right]. \tag{1.0.1}$$

Let

$$S_X^2(n) \equiv \frac{1}{n-1}\sum_{i=1}^{n}[X_i - \overline{X}(n)]^2 \tag{1.0.2}$$

be the sample variance of $X_1, \ldots, X_n$. One has

$$\mathrm{E}(S_X^2(n)) = \sigma_X^2\left[1 - \frac{2}{n-1}\sum_{i=1}^{n-1}\left(1 - \frac{i}{n}\right)\rho_i\right] \tag{1.0.3}$$

(cf. Anderson [5]) and

$$\mathrm{E}\left(\frac{S_X^2(n)}{n}\right) = \left[\frac{(n/a_n)-1}{n-1}\right]\mathrm{Var}(\overline{X}(n)), \tag{1.0.4}$$

where $a_n \equiv 1 + 2\sum_{i=1}^{n-1}(1-i/n)\rho_i$.

Suppose momentarily that the X_i's are independent. Equation (1.0.3) implies that $S_X^2(n)$ is an unbiased estimator of the population variance σ_X^2, and Equation (1.0.4) implies that $S_X^2(n)/n$ is an unbiased estimator of $\mathrm{Var}(\overline{X}(n))$. Unfortunately, simulation output processes are typically positively correlated, that is, $\rho_i > 0$, $i \geq 1$. In this case, Equations (1.0.3) and (1.0.4) imply that $\mathrm{E}(S_X^2(n)) < \sigma_X^2$ and $\mathrm{E}(S_X^2(n)/n) < \mathrm{Var}(\overline{X}(n))$, respectively. The last inequality indicates that the "classical" confidence interval for μ_X

$$\overline{X}(n) \pm z_{1-\alpha/2}\frac{S_X(n)}{\sqrt{n}},$$

where $z_{1-\alpha}$ is the $1-\alpha$ quantile of the $\mathrm{Nor}(0,1)$ distribution, will contain the unknown μ_X with a probability that is considerably less than the nominal value of $1-\alpha$.

Recall that $\overline{X}(n)$ is a consistent estimator of μ_X if $\mathrm{Var}(\overline{X}(n)) \to 0$ as $n \to \infty$ or $\lim_{n\to\infty}\sigma_n^2 < \infty$, where $\sigma_n^2 \equiv n\mathrm{Var}(\overline{X}(n))$ for every $n \geq 1$. By Equation (1.0.1), the last condition is equivalent to

$$\lim_{n\to\infty}\sum_{i=1}^{n-1}\left(1-\frac{i}{n}\right)R_i < \infty. \tag{1.0.5}$$

A necessary and sufficient condition for (1.0.5) is

$$\sum_{i=-\infty}^{\infty}|R_i| < \infty, \tag{1.0.6}$$

whence

$$\sigma^2 \equiv \lim_{n\to\infty}\sigma_n^2 = \sum_{i=-\infty}^{\infty}R_i \tag{1.0.7}$$

(cf. Anderson [5]). We call σ^2 the *variance parameter* of the process $\{X_i, i \geq 1\}$.

Remark 1.0.1 If $\rho_i > 0\ \forall i$ and $\sigma^2 < \infty$, then

$$\mathrm{E}(S_X^2(n)) < \sigma_X^2 < \sigma_n^2 \to \sigma^2 \quad \text{as } n \to \infty. \quad \triangleleft$$

While the estimation the variance of a process of independent random variables is a trivial problem, the estimation of the variance parameter of a correlated stationary stochastic process is a challenging problem. In fact, for highly correlated processes, the variance parameter may not even exist. One should start by imposing conditions on the structure of the process to be able to guarantee the existence of such a parameter. With existence in hand, different approaches have been used to estimate the variance parameter. Some of them divide a long simulation run into batches with the final objective of reducing the estimation problem to the independent case—so that classic estimation techniques can be used. The downside is that these estimators are biased. Another approach is to allow a standardized time series (STS) to converge to a Brownian bridge and consider (i) the square of the weighted area under this Brownian bridge (weighted area estimator); or (ii) the weighted area under the square of the Brownian bridge (Cramér-von Mises estimator). Intuitively, since these areas have expectation σ^2, the discrete version of such integrals should behave like "good" estimators for σ^2. These STS-based estimators have bias and variance that depend upon the weight function, a fact which allows us to choose these functions adequately so that the corresponding estimators are "asymptotically unbiased" and have variance comparable to that of existing estimators from the literature.

In this thesis we aim to answer the following questions:

- Can we generalize the STS-based estimators to a wider class of estimators without losing their original properties?
- Could we linearly combine several of these estimators to obtained even "better" ones?
- Will these estimators be highly correlated or not?
- Would we be able to apply batching techniques to the new estimators without losing their original properties?
- Will we be able to implement these new estimators efficiently?

We will answer these questions by studying a new way to obtain Brownian bridges from a standard Brownian bridge on $[0, 1]$. We call this methodology "folded Brownian bridges".

The operation of "folding" a Brownian bridge consists of reflecting a standard Brownian bridge through the line $t = 1/2$ and stretching it so that it spans the interval $[0, 1]$. Then, we cross our fingers hoping that folding the original STS will produce a new STS converging to a new Brownian bridge. Applying this transformation a number of times to the original Brownian bridge and STS, we create a sequence of estimators (one for each fold) called folded estimators.

Our estimators have reasonable theoretical significance since not only will they generalize already existing variance estimators, but they will be asymptotically unbiased and, by combining them carefully, will have comparatively lower variance than their original counterparts. Moreover, they can be used in practice in numerous areas such as quality control, queuing theory, process control, simulation output analysis of complex systems, etc. In queuing theory, for instance, the waiting times of a bank teller are an example of a highly correlated process. Moreover, good estimates of the mean waiting time should always be accompanied by good estimates for the variance parameter so that we can obtain reliable confidence intervals. The information derived from these intervals is what is used to make any changes in the system, with the final objective being the improvement of the system.

Several estimators for the variance parameter of a stochastic process have been studied in the literature. We review most of them in the remainder of this chapter. Section 1.1 reviews the assumptions imposed on the underlying stochastic process for the variance parameter to exist as well as some basic concepts needed in the sequel. In Section 1.2 we describe some of the estimators in the literature and their main properties. In the last section we present an outline of the remaining chapters.

1.1 Basic Concepts and Assumptions

Throughout, we consider a stationary time series $\{X_i,\ i \geq 1\}$ that satisfies a *Functional Central Limit Theorem* (FCLT) Assumption. This assumption applies to a broad class of processes, and will allow us to determine the limiting properties of the various variance estimators considered herein.

Assumption FCLT There exist finite constants μ and $\sigma > 0$ such that as $n \to \infty$,

$$\left\{ \frac{\lfloor nt \rfloor \left(\overline{X}_{\lfloor nt \rfloor} - \mu \right)}{\sqrt{n}} \right\}_{0 \le t \le 1} \Longrightarrow \{\sigma \mathcal{W}(t)\}_{0 \le t \le 1}, \tag{1.1.1}$$

where $\mathcal{W}$ is a standard Brownian motion process and $\lfloor \cdot \rfloor$ is the greatest integer function, and the convergence $\Longrightarrow$ in Equation (1.1.1) is weak convergence in the space $D\,[0,1]$ equipped with the Skorohod topology; see Billingsley [9, p. 153].

Remark 1.1.1 The sample paths of $\lfloor nt \rfloor \left(\overline{X}_{\lfloor nt \rfloor} - \mu \right)/\sqrt{n}$ lie in $D\,[0,1]$, the space of functions on $[0,1]$ that are right-continuous and have left-hand limits, while the sample paths of $\mathcal{W}$ lie in $C\,[0,1]$, the space of continuous functions on $[0,1]$. $\triangleleft$

Remark 1.1.2 Glynn and Iglehart [21] list several sufficient conditions for Assumption FCLT to hold. In most cases the constants μ and σ^2 in the assumption are the process mean μ_X and the variance parameter σ^2, respectively. For this reason, we will study estimation techniques for the variance parameter. $\triangleleft$

The standardized time series (STS) of the stochastic process $\{X_i, i \ge 1\}$ is defined (see Schruben [42]) as follows:

$$T_n(t) \equiv \frac{\lfloor nt \rfloor \left(\overline{X}_n - \overline{X}_{\lfloor nt \rfloor} \right)}{\sigma \sqrt{n}}, \quad \text{for } 0 \le t \le 1. \tag{1.1.2}$$

Remark 1.1.3 Under Assumption FCLT, it can be shown that, as $n \to \infty$,

$$\left(\sqrt{n} \left(\overline{X}_n - \mu \right), \sigma T_n \right) \Longrightarrow (\sigma \mathcal{W}(1), \sigma \mathcal{B}), \tag{1.1.3}$$

where $\mathcal{B}$ is the standard *Brownian bridge* process on $[0,1]$ associated with the Brownian Motion $\mathcal{W}$; that is, $\mathcal{B}(t) \equiv t\mathcal{W}(1) - \mathcal{W}(t)$. See Glynn and Iglehart [21], Foley and Goldsman [20], or Schruben [42] for the proof and further details. Recall that all finite-dimensional distributions of $\mathcal{B}$ are jointly normal with $\mathrm{E}(\mathcal{B}(t)) = 0$ and $\mathrm{Cov}(\mathcal{B}(t), \mathcal{B}(s)) = \min(s,t) - st$, for $0 < s, t < 1$. Further, notice that $\mathcal{W}(1)$ and $\mathcal{B}$ are independent. Three additional useful properties fall out of Equation (1.1.3):

- $\sqrt{n} \left(\overline{X}_n - \mu \right)$ is asymptotically $\sigma\mathrm{Nor}(0,1)$,

- σT_n is asymptotically σ times a Brownian bridge, and
- $\sqrt{n}\left(\overline{X}_n - \mu\right)$ and σT_n are asymptotically independent; thus all information gleaned from σT_n will be asymptotically independent of $\sqrt{n}\left(\overline{X}_n - \mu\right)$. $\triangleleft$

Mixing Processes Let $\{X_i, i \geq 1\}$ be a stationary sequence of random variables defined on a probability space $(\Omega, \mathbb{B}, P)$. For $a \leq b$, define $\mathcal{M}_a^b$ as the sigma-field generated by $X_a, \ldots, X_b$; define $\mathcal{M}_1^a$ as the sigma-field generated by $X_1, \ldots, X_a$; and define $\mathcal{M}_a^\infty$ as the sigma-field generated by $X_a, X_{a+1}, \ldots$. Now, consider a nonnegative function ϕ defined over positive integers. We shall say that the sequence $\{X_i, i \geq 1\}$ is ϕ-*mixing* if for each positive integer k and for each n $(n \geq 1)$, $E_1 \in \mathcal{M}_1^k$ and $E_2 \in \mathcal{M}_{k+n}^\infty$ together imply

$$|P(E_1 \cap E_2) - P(E_1)P(E_2)| \leq \phi(n)P(E_1). \tag{1.1.4}$$

This is a joint property of $\{X_i, i \geq 1\}$ and ϕ. We consider only functions ϕ satisfying

$$\lim_{n\to\infty} \phi(n) = 0, \tag{1.1.5}$$

and usually we require that $\phi(n)$ goes to 0 at some specified minimum rate. If we say that $\{X_i, i \geq 1\}$ is ϕ-*mixing* without specifying ϕ, we mean that Property (1.1.4) holds for some ϕ satisfying Equation (1.1.5). Roughly speaking, in a ϕ-*mixing* process, the distant future is virtually independent of the past and present. See Billingsley [9, p. 166] for further details.

Uniform Integrability The random sequence $\{X_i, i \geq 1\}$ defined on a probability space $(\Omega, \mathbb{B}, P)$ is said to be *uniformly integrable* if

$$\lim_{\alpha\to\infty} \sup_i \int_{\{|X_i|\geq\alpha\}} |X_i|\, dP \equiv \mathrm{E}(|X_i| \mathbb{1}_{\{|X_i|\geq\alpha\}}) = 0,$$

where $\mathbb{1}_{\{|X_i|\geq\alpha\}}$ is the indicator function of the set $\{|X_i| \geq \alpha\}$. Also, notice that if the $\{X_i, i \geq 1\}$ are uniformly integrable, then

$$\sup_i \mathrm{E}(|X_i|) < \infty.$$

See Billingsley [9, p. 32] for more information regarding uniform integrability.

Section 1.2 motivates the importance of variance parameter estimation for stationary time series. It also reviews briefly some popular variance parameter estimation techniques.

1.2 Variance Estimators for Stationary Processes

This section reviews some of the methods in the literature for the estimation of the process variance parameter, $\sigma^2 \equiv \lim_{n\to\infty} n\mathrm{Var}(\overline{X}(n))$, the measure of precision that supplements estimation of the population mean μ_X when computing confidence intervals for it. Notice that under Assumption FCLT (1.1.1), the existence of such a parameter is taken for granted.

1.2.1 Nonoverlapping Batch Means (NBM) Variance Estimator

Original accounts on the NBM method were given by Conway [12], Fishman [15, 16], and Law and Carson [30]. This method is commonly used to compute confidence intervals for the mean μ_X, and it owes its reputation to its simplicity and performance.

Again, let $\{X_i,\, 1 \le i \le n\}$ be the output of a long simulation run from a stationary process with a finite variance parameter σ^2. Split the output data into b adjacent, nonoverlapping batches, each consisting of m observations, where with little loss of generality we assume that $n = mb$. Therefore, the i^{th} batch consists of the observations

$$X_{(i-1)m+1}, X_{(i-1)m+2}, \ldots, X_{im}$$

for $i = 1, 2, \ldots, b$, and the i^{th} batch mean is the sample average of the observations from the i^{th} batch:

$$\overline{X}_{i,m} \equiv \frac{1}{m}\sum_{j=1}^{m} X_{(i-1)m+j}.$$

It can be shown (see Fishman [17] or Law and Carson [30]) that for large m,

$$\overline{X}_{1,m}, \overline{X}_{2,m}, \ldots, \overline{X}_{b,m} \overset{i.i.d.}{\approx} \mathrm{Nor}\left(\mu, \frac{\sigma^2}{m}\right),$$

where $\overset{i.i.d.}{\approx}$ means *asymptotically independent and identically distributed*. Under the stationarity assumption for $\{X_i,\, i \ge 1\}$, the batch means $\overline{X}_{1,m}, \overline{X}_{2,m}, \ldots, \overline{X}_{b,m}$ are identically distributed, but typically dependent for small values of m. The asymptotic normality of each batch mean follows from Assumption FCLT. The low correlation among the batch means for large m follows from Law and Carson [30] who showed that, as $m \to \infty$,

$$\mathrm{Corr}(\overline{X}_{1,m}, \overline{X}_{1+i,m}) \to 0 \quad \forall i \geq 1. \tag{1.2.1}$$

See also Fishman [17] and Alexopoulos and Goldsman [2] for additional details.

Finally, the NBM estimator for σ^2 is defined by

$$\widehat{V}_B \equiv \frac{m}{b-1} \sum_{i=1}^{b} \left(\overline{X}_{i,m} - \overline{X}(n)\right)^2. \tag{1.2.2}$$

The next result gives some expressions for the expected value of the NBM estimator for σ^2. In addition, we assume $\sum_{j=1}^{\infty} j|R_j| < \infty$ and we define the related constant

$$\gamma \equiv -2 \sum_{i=1}^{\infty} jR_j \tag{1.2.3}$$

from Song and Schmeiser in [44]. At this point we define the "little-oh" and "big-Oh" notations. We say that $f(m) = \mathrm{o}(g(m))$ if $f(m)/g(m) \to 0$ as $m \to \infty$. Additionally, we say that $f(m) = \mathrm{O}(g(m))$ if there exists a positive constant C and a non-negative integer m_0 such that $|f(m)/g(m)| \leq C$ for every $m \geq m_0$.

Theorem 1.2.1 (Goldsman and Meketon [25]; Song and Schmeiser [44]) *If $\{X_i, i \geq 1\}$ is stationary with $\mathrm{E}(X_1^4) < \infty$ and ϕ-mixing with $\phi(k) = \mathrm{O}(k^{-4-\epsilon})$ for some $\epsilon > 0$, then $\sum_{j=1}^{\infty} j|R_j| < \infty$ and*

$$\mathrm{E}(\widehat{V}_B) = \sigma^2 + \frac{\gamma\,(b+1)}{n} + \mathrm{o}(1/m), \tag{1.2.4}$$

where γ is the constant defined in Equation (1.2.3).

Since the batch means are identically distributed but not necessarily independent, additional conditions on the process will have to be imposed before we can state variance and distributional results for the batch means variance estimator $\widehat{V}_B$.

Theorem 1.2.2 (Goldsman and Meketon [25]; Song and Schmeiser [44]; Chien, Goldsman, and Melamed [11]) *If $\{X_i, i \geq 1\}$ is stationary with $\mathrm{E}(X_1^{12}) < \infty$ and ϕ-mixing with $\phi(k) = \mathrm{O}(k^{-9})$, then*

$$b\mathrm{Var}(\widehat{V}_B) = \frac{2\sigma^4 b(b+1)}{(b-1)^2} + \mathrm{O}(m^{-1/4}) + \mathrm{O}(1/b) = 2\sigma^4 + \mathrm{o}(1), \tag{1.2.5}$$

the last equality holding as $m \to \infty$ and $b \to \infty$.

For b fixed, different, but still mild, moment and mixing conditions imply (see Glynn and Whitt [22])

$$\widehat{V}_B \xrightarrow{\mathcal{D}} \frac{\sigma^2}{b-1}\chi^2(b-1), \tag{1.2.6}$$

as $m \to \infty$, where $\xrightarrow{\mathcal{D}}$ stands for convergence in distribution, and $\chi^2(b-1)$ denotes a χ^2 random variable with $b-1$ degrees of freedom. Additionally, uniform integrability yields $\mathrm{E}(\widehat{V}_B) \to \sigma^2$ and $\mathrm{Var}(\widehat{V}_B) \to 2\sigma^4/(b-1)$ as $m \to \infty$. Interesting tradeoffs arise when deciding how to choose the batch size (and the number of batches) since as b increases, $\mathrm{Var}(\widehat{V}_B)$ decreases, while $\mathrm{Bias}(\widehat{V}_B) = \mathrm{E}(\widehat{V}_B) - \sigma^2$ decreases as m increases. See, for example, Fishman [18] and Schmeiser [40], among others.

1.2.2 Overlapping Batch Means (OBM) Variance Estimator

A variation of the traditional batch means technique is the OBM method proposed by Meketon and Schmeiser [34]. For a given batch size m, the batches are built in the following way:

$$\begin{array}{cccccccc}
X_1 & X_2 & \ldots & X_m & & & & \\
 & X_2 & X_3 & \ldots & X_{m+1} & & & \\
 & & \ddots & & & \ddots & & \\
 & & & X_{n-m} & \ldots & X_{n-1} & X_{n-1} & \\
 & & & & X_{n-m+1} & \ldots & X_{n-1} & X_n
\end{array}$$

The i^{th} overlapping batch mean is the sample average of the i^{th} batch:

$$\overline{X}(i,m) \equiv \frac{1}{m}\sum_{j=0}^{m-1} X_{i+j}, \tag{1.2.7}$$

for $i = 1, 2, \ldots, n-m+1$. The OBM estimator for σ^2 is

$$\widehat{V}_O \equiv \frac{nm}{(n-m+1)(n-m)} \sum_{i=1}^{n-m+1} \left(\overline{X}(i,m) - \overline{X}(n)\right)^2. \tag{1.2.8}$$

Notice that $\overline{X}(1,m), \overline{X}(2,m), \ldots, \overline{X}(n-m+1,m)$ are not independent. However, they are identically distributed and become approximately normal as m increases; in fact, no attempt is made to make the overlaping batches independent. Moreover, under some mild moment and ϕ-mixing conditions, it can be shown (see Sargent, et al. [39]) that

$$\widehat{V}_O \approx \frac{\sigma^2 \chi^2\left(\frac{3}{2}(b-1)\right)}{\frac{3}{2}(b-1)}. \tag{1.2.9}$$

Under similar conditions (see Goldsman and Meketon [25]), one can show

$$\mathrm{E}(\widehat{V}_O) = \sigma^2 + \frac{\gamma}{n} + \mathrm{o}(1/m), \tag{1.2.10}$$

where γ is defined in (1.2.3), and (see Meketon and Schmeiser [34]),

$$\mathrm{Var}(\widehat{V}_O) = \frac{4\sigma^4}{3b} + \mathrm{o}(1/b). \tag{1.2.11}$$

Finally, while $\widehat{V}_B$ and $\widehat{V}_O$ are both asymptotically unbiased estimators of σ^2, we see that $\mathrm{Var}(\widehat{V}_O)$ is asymptotically smaller than $\mathrm{Var}(\widehat{V}_B)$. Indeed,

$$\frac{\mathrm{Var}(\widehat{V}_O)}{\mathrm{Var}(\widehat{V}_B)} \to \frac{2}{3}$$

as $m \to \infty$ and $b \to \infty$. See Song and Schmeiser [44] or Pedrosa and Schmeiser [37] for discussions on the choices of b and m.

1.2.3 The Weighted Area Estimator

This subsection deals with the standardized time series (STS) weighted area estimator for σ^2. An unweighted version of this estimator was first proposed by Schruben [42]. Further details can be found in Goldsman and Meketon [25], and Goldsman, Meketon, and Schruben [26].

In the sequel, we will require a number of assumptions to hold. We place them in an itemized list, and invoke them whenever they are needed.

Assumptions A

1. The process $\{X_i, i \geq 1\}$ is stationary.

2. The process $\{X_i, i \geq 1\}$ satisfies Assumption FCLT.

3. $\sum_{k=-\infty}^{\infty} R_k = \sigma^2 > 0$.

4. $\sum_{k=1}^{\infty} k^2 |R_k| < \infty$.

5. $w(\cdot)$ is a function defined on $[-1/n, 1+1/n]$ such that $w''(t)$ is continuous and bounded on $[-1/n, 1+1/n]$ for some $n \geq 1$, and $2\int_0^1 \int_0^t w(s)w(t)s(1-t)\,ds\,dt = 1$ (a normalizing assumption).

Remark 1.2.3 Assumptions A.1–A.4 are conditions on the underlying stochastic process. Assumptions A.3 and A.4 hold for a variety of stochastic processes. Assumption A.5 is simply a set of conditions on a weight function $w(\cdot)$ that will be used later on in the definition of some estimators. $\triangleleft$

We begin by defining the square of the weighted area under an STS and the square of the weighted area under a standard Brownian bridge.

Definition 1.2.4 *The square of the weighted area under an STS is defined by*

$$A(w,n) \equiv \left[\frac{1}{n}\sum_{k=1}^{n} w\left(\frac{k}{n}\right)\sigma T_n\left(\frac{k}{n}\right)\right]^2, \tag{1.2.12}$$

where $T_n(\cdot)$ is defined by (1.1.2); and the square of the weighted area under a standard Brownian bridge $\mathcal{B}(t)$ is defined by

$$A(w) \equiv \left[\int_0^1 w(t)\sigma\mathcal{B}(t)\,dt\right]^2, \tag{1.2.13}$$

where in both cases $w(\cdot)$ is a weight function satisfying Assumption A.5.

Under Assumptions FCLT and A.5, the continuous mapping theorem (Theorem 5.5 of [9, p. 34]) implies that $A(w,n) \xrightarrow{\mathcal{D}} A(w) \sim \sigma^2\chi_1^2$ as $n \to \infty$. For this reason, we call $A(w,n)$ the *weighted area estimator for* σ^2. The next theorem gives expressions for the expected value and variance of the weighted area estimator.

Theorem 1.2.5 (Goldsman, Meketon, and Schruben [26]) *Suppose Assumptions A hold. Further, suppose that the sequence $\{A^2(w,n), n \geq 1\}$ is uniformly integrable. Then*

$$\mathrm{E}(A(w,n)) = \sigma^2 + \frac{\left[\left(W - \overline{W}\right)^2 + \overline{W}^2\right]\gamma}{2n} + \mathrm{o}(1/n) \tag{1.2.14}$$

and

$$\mathrm{Var}(A(w,n)) \to \mathrm{Var}(A(w)) = \mathrm{Var}(\sigma^2\chi_1^2) = 2\sigma^4 \quad \textit{as } n \to \infty, \tag{1.2.15}$$

where $W(s) \equiv \int_0^s w(t)\,dt$, for $0 \leq s \leq 1$, $W \equiv W(1)$, and $\overline{W} \equiv \int_0^1 W(s)\,ds$.

Notice that the limiting variance does not depend on the form of the weight function. The following examples review various weight functions.

Example 1.2.6 The area estimator with constant weight function $w_0(t) \equiv \sqrt{12}$ for all $t \in [0,1]$ yields $\mathrm{E}(A(w_0,n)) = \sigma^2 + \frac{3\gamma}{n} + \mathrm{o}(1/n)$ (see Schruben [42]). $\triangleleft$

Example 1.2.7 If one chooses weights satisfying $W = \overline{W} = 0$, the resulting estimator is *first-order unbiased* for σ^2, i.e., its bias is $\mathrm{o}(1/n)$. Such a weight function is $w_2(t) \equiv \sqrt{840}\left(3t^2 - 3t + 1/2\right)$ (see [26] and [27]). $\triangleleft$

Example 1.2.8 Other weight functions yielding first-order unbiased estimators for σ^2 are given by the family $w_{\cos,j}(t) = \sqrt{8}\pi j \cos(2\pi j t)$, $j \geq 1$. Foley and Goldsman [20] show that this orthonormal sequence of weights produces variance estimators $A(w_{\cos,1},n)$, $A(w_{\cos,2},n)$, $\ldots$ that are not only first-order unbiased, but also asymptotically independent; that is, $A(w_{\cos,1},n)$, $A(w_{\cos,2},n), \ldots$ are i.i.d. $\sigma^2\chi^2(1)$ as $n \to \infty$. $\triangleleft$

1.2.4 The Weighted Cramér-von Mises (CvM) Estimator

This section gives an overview of the weighted CvM estimator for σ^2. In the sequel, we will require another assumption to hold.

Assumption A.6 $g(\cdot)$ is a function defined on $[0,1]$ such that $g''(t)$ is continuous and bounded on $[0,1]$, and $\int_0^1 g(t)t(1-t)\,dt = 1$ (a normalizing assumption).

We begin by defining the weighted area under the square of an STS and the weighted area under the square of a standard Brownian bridge.

Definition 1.2.9 *The weighted area under the square of an STS is defined by*

$$C(g,n) \equiv \frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\sigma T_n\left(\frac{k}{n}\right)\right]^2, \tag{1.2.16}$$

where $T_n(\cdot)$ is defined by (1.1.2); and the weighted area under the square of a standard Brownian bridge $\mathcal{B}(t)$ is defined by

$$C(g) \equiv \int_0^1 g(t)[\sigma\mathcal{B}(t)]^2\,dt. \tag{1.2.17}$$

Under Assumptions FCLT and A.6, the continuous mapping theorem (Theorem 5.5 of [9, p. 34]) implies that $C(g,n) \xrightarrow{\mathcal{D}} C(g)$ as $n \to \infty$. Since $C(g,n)$ resembles a Cramér-von Mises statistic, we call $C(g,n)$ the *weighted CvM estimator* for σ^2.

The next theorem gives results on the expected value and variance of the weighted CvM estimator.

Theorem 1.2.10 (Goldsman, Kang and Seila [23]) *Suppose Assumptions A.1–A.4 and A.6 hold. Further, suppose that the sequence $\{C^2(g,n), n \ge 1\}$ is uniformly integrable. Then*

$$\mathrm{E}(C(g,n)) = \sigma^2 + \frac{\gamma}{n}(G-1) + \mathrm{o}(1/n) \tag{1.2.18}$$

and

$$\mathrm{Var}(C(g,n)) \to \mathrm{Var}(C(g)) = 4\sigma^4 \int_0^1 g(t)(1-t)^2 \int_0^t g(s)s^2\,ds\,dt \quad \text{as } n \to \infty, \tag{1.2.19}$$

where $G \equiv \int_0^1 g(t)\,dt$.

Notice that the limiting variance depends on the form of the weight function g. For comparisons, let us consider the following examples:

Example 1.2.11 The CvM estimator with constant weight function $g_0(t) \equiv 6$ yields $\mathrm{E}(C(g_0,n)) = \sigma^2 + 5\gamma/n + \mathrm{o}(1/n)$ and $\mathrm{Var}(C(g_0)) = 4\sigma^4/5$. $\triangleleft$

If one chooses weights having $G = 1$, in addition to the normalizing and second derivative constraints imposed on g, Theorem 1.2.10 implies that the CvM estimator is first-order unbiased.

Example 1.2.12 Consider $g_{2,c}(t) \equiv 51 - c/2 + ct - 150t^2$, where $t \in [0,1]$ and c is a real constant. Besides being first-order unbiased, the CvM estimator with weight function $g_{2,c}(t)$ satisfies

$$\mathrm{Var}(C(g_{2,c})) = \frac{(c^2 - 300c + 26856)\,\sigma^4}{2520}.$$

This variance is minimized by $g_2^*(t) = g_{2,150}(t)$ whence

$$g_2^*(t) = -24 + 150t - 150t^2$$

and

$$\mathrm{Var}(C(g_2^*)) = \frac{121\sigma^4}{70} > \mathrm{Var}(C(g_0)). \quad \triangleleft$$

Example 1.2.13 First-order unbiased minimum-variance polynomial weights with degrees of 4 and 6 are derived in [23]. These are

$$g_4^*(t) = \frac{-1310}{21} + \frac{19270t}{21} - \frac{25230t^2}{7} + \frac{16120t^3}{3} - \frac{8060t^4}{3}$$

with $\mathrm{Var}(C(g_4^*)) = 1.042\sigma^4$; and the sixth-degree polynomial is given by

$$g_6^*(t) = \sum_{i=0}^{6} c_i t^i,$$

where

$$\begin{aligned} c_0 &= -132.9358, & c_1 &= 3439.9542, & c_2 &= -26622.7987, & c_3 &= 93037.7083, \\ c_4 &= -163198.9022, & c_5 &= 140016.0576, & c_6 &= -46672.0191, \end{aligned}$$

and $\mathrm{Var}(C(g_6^*)) = 0.8093\sigma^4$. $\triangleleft$

The STS area and CvM estimators from Definitions 1.2.4 and 1.2.9 are based on one batch of observations. One can apply batching to the observations $X_1, X_2, \ldots, X_n$, calculate an STS estimator from each batch, and eventually average those estimators over the b batches. These batched estimators turned out to have smaller variance than those based on one long batch. See Alexopoulos et al. [1] and Chapter VI of the current thesis for further details. In the upcoming chapters, we will present a different approach by applying "folding" techniques to the already familiar STS estimators.

1.3 Outline of the Remaining Chapters

After having discussed some of the popular techniques developed for the estimation of the variance parameter of a stationary time series, we briefly describe the scope and original contributions of this research and outline the rest of the chapters.

This thesis is concerned with the development of new variance estimators for the analysis of simulation output, in particular, estimators for the variance of the sample mean $\overline{X}(n)$.

Knowledge about the variance of the sample mean is useful for constructing valid confidence intervals for the mean of stationary stochastic processes. This variance estimation problem is challenging in a simulation environment, where observations are rarely independent, identically distributed normal random variables. The upshot is that one cannot use "standard" statistical techniques to estimate the variance of the sample mean or to construct confidence intervals for the true mean. We resort to the use of STS to develop and study a new class of variance estimators for stationary stochastic processes. Our estimators are generalizations of Schruben's weighted area and weighted CvM estimators described in Chapter I.

The thesis is organized in the following way. Chapter II introduces the concept of folding a Brownian bridge, and shows that a sequence of successive folding actions induces Brownian bridges at various levels. Then we derive useful expressions for these Brownian bridges in terms of the original (level-0) Brownian bridge and in terms of the original standard Brownian motion. We use these formulas later on in the analysis of two new variance parameter estimators. The second part of Chapter II proves that the unweighted areas under the different levels of folding of a Brownian bridge form an i.i.d. sequence of $\text{Nor}(0,1)$ random variables. Chapter III introduces and studies the asymptotic properties of the folded version of the weighted area estimator. We propose several families of weight functions yielding first-order unbiased estimators, and we show that the limiting variance of the respective variance estimators does not depend upon the weight function. Chapter IV contains a parallel analysis to the one presented in Chapter III, this time for the folded version of the CvM estimator. Since in this case the limiting variance will depend on the choice of the weight function, we compute it for different weight functions yielding first-order unbiased estimators. Chapter V contains experimental results obtained from a Monte Carlo performance evaluation of the estimators formulated in this research. Chapter VI discusses batched versions of the folded estimators from Chapters III and IV. Chapter VII lists the conclusions and directions for further investigation. The Appendices contain the proofs of the main results.

CHAPTER II

FOLDED BROWNIAN BRIDGES AND THEIR FOLDED STANDARDIZED TIME SERIES

This chapter is organized in three parts. Section 2.1 introduces the definition of the different *folded* levels of a Brownian bridge and shows that they are themselves Brownian bridges. We also derive useful expressions for them in terms of the original (level-0) Brownian bridge and in terms of the original standard Brownian motion. We use these formulas later on in the analysis of two new variance parameter estimators. In Section 2.2, we show that the sequence of unweighted areas under the different levels of a Brownian bridge is i.i.d. $\mathrm{Nor}(0, 1)$. Finally, in Section 2.3 we define the different folded levels of an STS and derive expressions for them in terms of the original (level-0) STS and in terms of the underlying stochastic process $\{X_i, i \geq 1\}$.

2.1 Definition and Properties of Folded Brownian Bridges

Recall that the usual Brownian bridge process is defined by

$$\mathcal{B}(t) \equiv \mathcal{B}_0(t) \equiv \mathcal{W}(t) - t\mathcal{W}(1) \quad \text{for } 0 \leq t \leq 1, \tag{2.1.1}$$

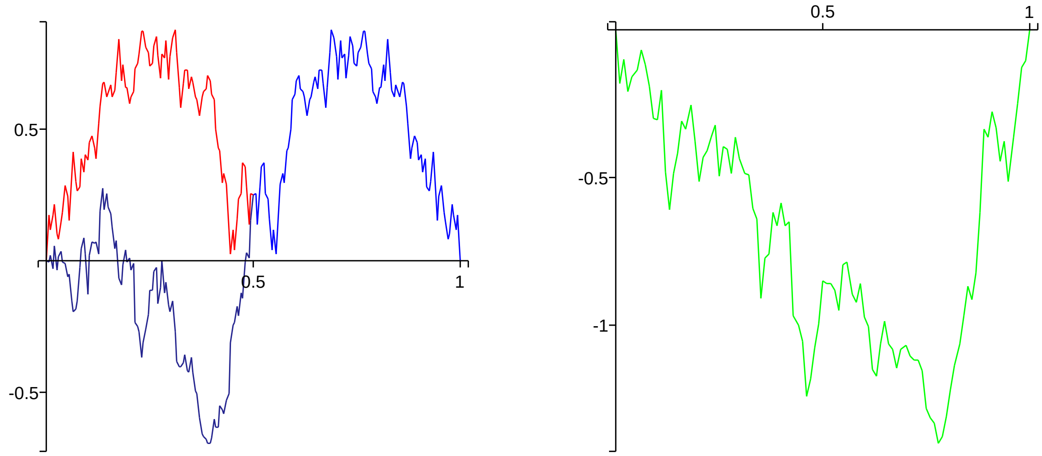
where $\mathcal{W}(\cdot)$ is the standard Brownian motion process.

Remark 2.1.1 Observe that Equation (2.1.1) represents a Brownian bridge as a Gaussian process (since $\mathcal{W}(\cdot)$ is Gaussian) that satisfies $\mathrm{E}(\mathcal{B}(s)) = 0$ for $s < 1$ and $\mathrm{Cov}(\mathcal{B}(s), \mathcal{B}(t)) = s(1-t)$ for $0 < s < t < 1$. $\triangleleft$

Definition 2.1.2 *As in Shorack and Wellner [43], we define the level-*1 *Brownian bridge by*

$$\mathcal{B}_1(t) \equiv \mathcal{B}_0\left(\frac{t}{2}\right) - \mathcal{B}_0\left(1 - \frac{t}{2}\right) \quad \textit{for } 0 \leq t \leq 1.$$

Intuitively speaking, what Definition 2.1.2 means is that we (i) take a Brownian bridge (shown in blue in Figure 1.a) and reflect the portion after $t = 1/2$ (in light blue) through $t = 1/2$ (in red), and (ii) take the difference between the two portions and stretch it over the $[0, 1]$ interval (shown in Figure 1.b, in green). Note that in this particular example we ended up with a Brownian bridge that was almost always negative in the $[0, 1]$ interval, but this will not be the case in general.



a. Reflecting about $t = 1/2$ **b.** Differencing

Figure 1: Geometric Illustration of Folded Brownian Bridges.

Remark 2.1.3 Observe that $\mathcal{B}_1(t)$ is also a Brownian bridge. To show this, we first note that $\mathcal{B}_1(0) = \mathcal{B}_1(1) = 0$, and that $\mathcal{B}_1(t)$ is a Gaussian process. Now, we just need to show that $\mathcal{B}_1(t)$ has the appropriate covariance structure. Suppose that $0 < s < t < 1$. Then

$$\begin{aligned}
\mathrm{Cov}(\mathcal{B}_1(s), \mathcal{B}_1(t)) &= \mathrm{Cov}\left(\mathcal{B}_0\left(\frac{s}{2}\right) - \mathcal{B}_0\left(1 - \frac{s}{2}\right), \mathcal{B}_0\left(\frac{t}{2}\right) - \mathcal{B}_0\left(1 - \frac{t}{2}\right)\right) \\
&= \mathrm{Cov}\left(\mathcal{B}_0\left(\frac{s}{2}\right), \mathcal{B}_0\left(\frac{t}{2}\right)\right) - \mathrm{Cov}\left(\mathcal{B}_0\left(\frac{s}{2}\right), \mathcal{B}_0\left(1 - \frac{t}{2}\right)\right) \\
&\quad - \mathrm{Cov}\left(\mathcal{B}_0\left(1 - \frac{s}{2}\right), \mathcal{B}_0\left(\frac{t}{2}\right)\right) + \mathrm{Cov}\left(\mathcal{B}_0\left(1 - \frac{s}{2}\right), \mathcal{B}_0\left(1 - \frac{t}{2}\right)\right) \\
&= \left(\frac{s}{2} - \frac{s}{2} \cdot \frac{t}{2}\right) - \left(\frac{s}{2} - \frac{s}{2}\left(1 - \frac{t}{2}\right)\right) \\
&\qquad \text{(since } 0 < s < t < 1 \text{ implies } 0 < s/2 < t/2 < 1/2 \text{ and} \\
&\qquad 0 < s/2 < t/2 < 1/2 \text{ implies } s/2 < 1/2 < 1 - t/2)
\end{aligned}$$

$$
-\left(\frac{t}{2}-\frac{t}{2}\left(1-\frac{s}{2}\right)\right)+\left(\left(1-\frac{t}{2}\right)-\left(1-\frac{t}{2}\right)\left(1-\frac{s}{2}\right)\right)
$$

(since $s/2 < 1/2 < 1-t/2$ implies $t/2 < 1-s/2$ and

$t/2 < 1-s/2$ implies $1-t/2 < 1-s/2$)

$$
= s - st. \quad \triangleleft
$$

The last remark shows that as long as we start with a Brownian bridge, folding it will produce another Brownian bridge as well. This motivates the following definition.

Definition 2.1.4 *For $k \geq 1$, the level-k Brownian bridge is defined recursively by*

$$
\mathcal{B}_k(t) \equiv \mathcal{B}_{k-1}\left(\frac{t}{2}\right) - \mathcal{B}_{k-1}\left(1-\frac{t}{2}\right) \qquad \text{for } 0 \leq t \leq 1. \tag{2.1.2}
$$

The following lemma gives an equation relating the level-k Brownian bridge with the original (level-0) Brownian bridge and the initial Brownian motion process.

Lemma 2.1.5 *For $k \geq 1$,*

$$
\mathcal{B}_k(t) = \sum_{i=1}^{2^{k-1}} \left[\mathcal{B}\left(\frac{i-1}{2^{k-1}}+\frac{t}{2^k}\right) - \mathcal{B}\left(\frac{i}{2^{k-1}}-\frac{t}{2^k}\right)\right] \tag{2.1.3}
$$

$$
= \sum_{i=1}^{2^{k-1}} \left[\mathcal{W}\left(\frac{i-1}{2^{k-1}}+\frac{t}{2^k}\right) - \mathcal{W}\left(\frac{i}{2^{k-1}}-\frac{t}{2^k}\right)\right] + (1-t)\mathcal{W}(1). \tag{2.1.4}
$$

Proof: We will carry out the proof by induction. First, we express the level-2 bridge in terms of the level-0 Brownian bridge and the original Brownian motion process:

$$
\begin{aligned}
\mathcal{B}_2(t) &= \mathcal{B}_1\left(\frac{t}{2}\right) - \mathcal{B}_1\left(1-\frac{t}{2}\right) \\
&= \mathcal{B}_0\left(\frac{t}{4}\right) - \mathcal{B}_0\left(1-\frac{t}{4}\right) - \mathcal{B}_0\left(\frac{1}{2}-\frac{t}{4}\right) + \mathcal{B}_0\left(\frac{1}{2}+\frac{t}{4}\right) \\
&= \mathcal{W}\left(\frac{t}{4}\right) - \frac{t}{4}\mathcal{W}(1) - \mathcal{W}\left(1-\frac{t}{4}\right) + \left(1-\frac{t}{4}\right)\mathcal{W}(1) \\
&\quad - \mathcal{W}\left(\frac{1}{2}-\frac{t}{4}\right) + \left(\frac{1}{2}-\frac{t}{4}\right)\mathcal{W}(1) + \mathcal{W}\left(\frac{1}{2}+\frac{t}{4}\right) - \left(\frac{1}{2}+\frac{t}{4}\right)\mathcal{W}(1) \\
&= \mathcal{W}\left(\frac{t}{4}\right) - \mathcal{W}\left(1-\frac{t}{4}\right) - \mathcal{W}\left(\frac{1}{2}-\frac{t}{4}\right) + \mathcal{W}\left(\frac{1}{2}+\frac{t}{4}\right) + (1-t)\mathcal{W}(1) \\
&= \left[\mathcal{W}\left(\frac{t}{4}\right) - \mathcal{W}\left(\frac{1}{2}-\frac{t}{4}\right)\right] + \left[\mathcal{W}\left(\frac{1}{2}+\frac{t}{4}\right) - \mathcal{W}\left(1-\frac{t}{4}\right)\right] + (1-t)\mathcal{W}(1)
\end{aligned}
$$

thanks to Equation (2.1.1). Therefore, Equation (2.1.3) holds for $k = 2$.

Now, let us assume by the inductive hypothesis that, for all $0 \le t \le 1$,

$$\begin{aligned}
\mathcal{B}_{k-1}(t) &= \sum_{i=1}^{2^{k-2}} \left[\mathcal{B}\left(\frac{i-1}{2^{k-2}} + \frac{t}{2^{k-1}}\right) - \mathcal{B}\left(\frac{i}{2^{k-2}} - \frac{t}{2^{k-1}}\right) \right] \\
&= \sum_{i=1}^{2^{k-2}} \left[\mathcal{W}\left(\frac{i-1}{2^{k-2}} + \frac{t}{2^{k-1}}\right) - \mathcal{W}\left(\frac{i}{2^{k-2}} - \frac{t}{2^{k-1}}\right) \right] + (1-t)\mathcal{W}(1).
\end{aligned}$$

Then, using Definition 2.1.4, we have

$$\begin{aligned}
\mathcal{B}_k(t) &= \mathcal{B}_{k-1}\left(\frac{t}{2}\right) - \mathcal{B}_{k-1}\left(1 - \frac{t}{2}\right) \\
&= \sum_{i=1}^{2^{k-2}} \left[\mathcal{B}\left(\frac{i-1}{2^{k-2}} + \frac{t}{2^{k}}\right) - \mathcal{B}\left(\frac{i}{2^{k-2}} - \frac{t}{2^{k}}\right) \right] \\
&\qquad - \sum_{i=1}^{2^{k-2}} \left[\mathcal{B}\left(\frac{2i-1}{2^{k-1}} - \frac{t}{2^{k}}\right) - \mathcal{B}\left(\frac{2i-1}{2^{k-1}} + \frac{t}{2^{k}}\right) \right] \\
&\qquad \text{(by the inductive hypothesis)} \\
&= \sum_{i=1}^{2^{k-2}} \left[\mathcal{B}\left(\frac{2(i-1)}{2^{k-1}} + \frac{t}{2^{k}}\right) - \mathcal{B}\left(\frac{2i}{2^{k-1}} - \frac{t}{2^{k}}\right) \right] \\
&\qquad - \sum_{i=1}^{2^{k-2}} \left[\mathcal{B}\left(\frac{2i-1}{2^{k-1}} - \frac{t}{2^{k}}\right) - \mathcal{B}\left(\frac{2i-1}{2^{k-1}} + \frac{t}{2^{k}}\right) \right] \\
&= \sum_{i=1}^{2^{k-2}} \left[\mathcal{B}\left(\frac{2(i-1)}{2^{k-1}} + \frac{t}{2^{k}}\right) - \mathcal{B}\left(\frac{2i-1}{2^{k-1}} - \frac{t}{2^{k}}\right) \right] \\
&\qquad + \sum_{i=1}^{2^{k-2}} \left[\mathcal{B}\left(\frac{2i-1}{2^{k-1}} + \frac{t}{2^{k}}\right) - \mathcal{B}\left(\frac{2i}{2^{k-1}} - \frac{t}{2^{k}}\right) \right] \\
&\qquad \text{(rearranging the sums)} \\
&= \sum_{j \text{ odd}} \left[\mathcal{B}\left(\frac{j-1}{2^{k-1}} + \frac{t}{2^{k}}\right) - \mathcal{B}\left(\frac{j}{2^{k-1}} - \frac{t}{2^{k}}\right) \right] \\
&\qquad + \sum_{j \text{ even}} \left[\mathcal{B}\left(\frac{j-1}{2^{k-1}} + \frac{t}{2^{k}}\right) - \mathcal{B}\left(\frac{j}{2^{k-1}} - \frac{t}{2^{k}}\right) \right] \\
&\qquad \text{(taking } j = 2i-1 \ (1 \le j \le 2^{k-1}-1) \text{ in the first sum} \\
&\qquad \text{and } j = 2i \ (2 \le j \le 2^{k-1}) \text{ in the second sum)} \\
&= \sum_{j=1}^{2^{k-1}} \left[\mathcal{B}\left(\frac{j-1}{2^{k-1}} + \frac{t}{2^{k}}\right) - \mathcal{B}\left(\frac{j}{2^{k-1}} - \frac{t}{2^{k}}\right) \right] \quad \text{for } 0 \le t \le 1.
\end{aligned}$$

Equation (2.1.4) follows from applying Equation (2.1.1) to the last expression. □

2.2 The Joint Distribution of the Areas Under Unweighted Folded Brownian Bridges

Definition 2.2.1 *The area under the unweighted level-k Brownian bridge is defined as follows:*

$$N_k \equiv \sqrt{12} \int_0^1 \mathcal{B}_k\,(t)\; dt \quad \textit{for } k = 0, 1, \ldots. \tag{2.2.1}$$

where the $\sqrt{12}$ is a normalization constant so that N_k has variance one.

In Chapter III, we will show that N_k is in fact the limiting distribution of the folded version of Schruben's unweighted area estimator.

Next, we present a nice but unexpected result concerning the joint distribution of the N_k's.

Theorem 2.2.2 *The random variables $\{N_k, k \geq 0\}$ defined by (2.2.1) are i.i.d.* Nor$(0, 1)$.

Corollary 2.2.3 *If the random variables $\{N_k, k \geq 0\}$ are defined by (2.2.1), then*

$$\sum_{i=1}^{k} \frac{N_i^2}{k} \;\sim\; \frac{\chi^2(k)}{k}.$$

Proof: It follows directly from the definition of a $\chi^2(k)$ random variable and Theorem 2.2.2. $\Box$

To prove Theorem 2.2.2, we first need to prove a series of lemmas.

Lemma 2.2.4 $\mathrm{Cov}(N_0, N_k) = 0 \quad \forall k \geq 1.$

Proof: Indeed,

$$\begin{aligned}
\mathrm{Cov}(N_0, N_k) &= 12\,\mathrm{Cov}\left(\int_0^1 \mathcal{B}_0(s)\,ds, \int_0^1 \mathcal{B}_k(t)\,dt\right) \\
&= 12\mathrm{E}\left(\int_0^1 \mathcal{B}_0(s)\,ds \int_0^1 \mathcal{B}_k(t)\,dt\right) \\
&\qquad \text{(since } \mathrm{E}(\mathcal{B}_k(t)) = 0 \text{ for every } k \geq 0 \text{ and } 0 < t < 1) \\
&= 12\mathrm{E}\left(\int_0^1 \int_0^1 \mathcal{B}_0(s)\mathcal{B}_k(t)\,ds\,dt\right) \\
&= 12 \int_0^1 \int_0^1 \mathrm{E}(\mathcal{B}_0(s)\mathcal{B}_k(t))\,ds\,dt
\end{aligned}$$

(by Fubini's Theorem and the continuity of $\mathcal{B}_0(s)\mathcal{B}_k(t)$ on $[0,1]^2$)

$$= 12\int_0^1\int_0^1 \mathrm{Cov}\left[\mathcal{B}_0(s), \sum_{i=1}^{2^{k-1}}\left(\mathcal{B}_0\left(\frac{i-1}{2^{k-1}}+\frac{t}{2^k}\right)-\mathcal{B}_0\left(\frac{i}{2^{k-1}}-\frac{t}{2^k}\right)\right)\right] ds\,dt$$

(by Lemma 2.1.5 and $\mathrm{E}(\mathcal{B}_k(t)) = 0$ for every $k \geq 0$ and $0 < t < 1$).

$$= 12\int_0^1\int_0^1 \sum_{i=1}^{2^{k-1}} \mathrm{Cov}\left[\mathcal{B}_0(s), \mathcal{B}_0\left(\frac{i-1}{2^{k-1}}+\frac{t}{2^k}\right)-\mathcal{B}_0\left(\frac{i}{2^{k-1}}-\frac{t}{2^k}\right)\right] ds\,dt$$

(since the sum is finite, and the covariance is distributive)

$$= 12\int_0^1\int_0^1 \sum_{i=1}^{2^{k-1}} \mathrm{Cov}\left[\mathcal{B}_0(s), \mathcal{B}_0\left(\frac{i-1}{2^{k-1}}+\frac{t}{2^k}\right)\right] ds\,dt$$
$$- 12\int_0^1\int_0^1 \sum_{i=1}^{2^{k-1}} \mathrm{Cov}\left[\mathcal{B}_0(s), \mathcal{B}_0\left(\frac{2^{k-1}-i+1}{2^{k-1}}-\frac{t}{2^k}\right)\right] ds\,dt$$

(by symmetry with respect to i)

$$= 12\sum_{i=1}^{2^{k-1}}\left\{\int_0^1\int_0^1 \mathrm{Cov}\left[\mathcal{B}_0(s), \mathcal{B}_0\left(\frac{i-1}{2^{k-1}}+\frac{t}{2^k}\right)\right] ds\,dt\right.$$
$$\left.-\int_0^1\int_0^1 \mathrm{Cov}\left[\mathcal{B}_0(s), \mathcal{B}_0\left(1-\left(\frac{i-1}{2^{k-1}}+\frac{t}{2^k}\right)\right)\right] ds\,dt\right\}$$
$$= 12\sum_{i=1}^{2^{k-1}}\left\{\int_0^1\int_0^1 \mathrm{Cov}\left[\mathcal{B}_0(s), \mathcal{B}_0\left(\frac{i-1}{2^{k-1}}+\frac{t}{2^k}\right)\right] ds\,dt\right.$$
$$\left.-\int_0^1\int_0^1 \mathrm{Cov}\left[\mathcal{B}_0(1-s), \mathcal{B}_0\left(1-\left(\frac{i-1}{2^{k-1}}+\frac{t}{2^k}\right)\right)\right] ds\,dt\right\}$$

(by symmetry with respect to s)

$$= 0,$$

where the last equality follows from $\mathrm{Cov}\left(\mathcal{B}_0\left(a\right), \mathcal{B}_0\left(b\right)\right) = \mathrm{Cov}\left(\mathcal{B}_0\left(1-a\right), \mathcal{B}_0\left(1-b\right)\right)$. □

Lemma 2.2.5 *For $k \geq 1$, $j \geq 1$, $0 \leq s \leq 1$, and $0 \leq t \leq 1$,*

$$\mathrm{Cov}(\mathcal{B}_k(s), \mathcal{B}_{k+j}(t)) = \mathrm{Cov}(\mathcal{B}_0(s), \mathcal{B}_j(t)).$$

Proof: We proceed by induction on j. For $j = 1$ we have,

$$\begin{aligned}\mathrm{Cov}(\mathcal{B}_k(s), \mathcal{B}_{k+1}(t)) &= \mathrm{Cov}\left(\mathcal{B}_k(s), \mathcal{B}_k\left(\frac{t}{2}\right)-\mathcal{B}_k\left(1-\frac{t}{2}\right)\right)\\ &= \mathrm{Cov}\left(\mathcal{B}_0(s), \mathcal{B}_0\left(\frac{t}{2}\right)-\mathcal{B}_0\left(1-\frac{t}{2}\right)\right)\end{aligned}$$

$$\text{(the covariance structure of } \mathcal{B}_k(t) \text{ is the same } \forall k \geq 1)$$

$$= \mathrm{Cov}\left(\mathcal{B}_0(s), \mathcal{B}_1(t)\right) \quad \text{(by Definition 2.1.2).}$$

Next, as the inductive hypothesis, assume that for given $j \geq 1$, and any $k \geq 1$, $0 \leq s \leq 1$ and $0 \leq t \leq 1$, we have $\mathrm{Cov}(\mathcal{B}_k(s), \mathcal{B}_{k+j}(t)) = \mathrm{Cov}(\mathcal{B}_0(s), \mathcal{B}_j(t))$. By Definition 2.1.2,

$$\begin{aligned}
\mathrm{Cov}(\mathcal{B}_k(s), \mathcal{B}_{k+j+1}(t)) &= \mathrm{Cov}\left(\mathcal{B}_k(s), \mathcal{B}_{k+j}\left(\frac{t}{2}\right) - \mathcal{B}_{k+j}\left(1 - \frac{t}{2}\right)\right) \\
&= \mathrm{Cov}\left(\mathcal{B}_0(s), \mathcal{B}_j\left(\frac{t}{2}\right) - \mathcal{B}_j\left(1 - \frac{t}{2}\right)\right) \\
&\qquad \text{(by the inductive hypothesis)} \\
&= \mathrm{Cov}\left(\mathcal{B}_0(s), \mathcal{B}_{j+1}(t)\right). \quad \square
\end{aligned}$$

Lemma 2.2.6 *For $k \geq 0$, and $j \geq 1$,*

$$\mathrm{Cov}(N_k, N_{k+j}) = \mathrm{Cov}(N_0, N_j) = 0.$$

Proof:

$$\begin{aligned}
\mathrm{Cov}(N_k, N_{k+j}) &= 12\mathrm{Cov}\left(\int_0^1 \mathcal{B}_k(s)\, ds, \int_0^1 \mathcal{B}_{k+j}(t)\, dt\right) \\
&= 12 \int_0^1 \int_0^1 \mathrm{Cov}\left(\mathcal{B}_k(s), \mathcal{B}_{k+j}(t)\right) ds\, dt \\
&= 12 \int_0^1 \int_0^1 \mathrm{Cov}\left(\mathcal{B}_0(s), \mathcal{B}_j(t)\right) ds\, dt \\
&\qquad \text{(by Lemma 2.2.5)} \\
&= 12\mathrm{Cov}\left(\int_0^1 \mathcal{B}_0(s)\, ds, \int_0^1 \mathcal{B}_j(t)\, dt\right) \\
&= \mathrm{Cov}(N_0, N_j) \\
&= 0 \quad \text{(by Lemma 2.2.4).} \quad \square
\end{aligned}$$

Lemma 2.2.7 $N_k \sim \mathrm{Nor}(0,1)$ *for every* $k \geq 0$.

Proof: Since N_k is the integral of a continuous function over the closed inteval $[0, 1]$, its Riemann sum satisfies (see Bartle [8, p. 229])

$$L_k(m) \equiv \frac{\sqrt{12}}{m} \sum_{i=1}^{m} \mathcal{B}_k\left(\frac{i}{m}\right) \xrightarrow{\text{a.s}} N_k \quad \text{as } m \to \infty. \tag{2.2.2}$$

For fixed k and m, $L_k(m)$ has the normal distribution (as a finite linear combination of jointly normal random variables) with $\mathrm{E}(L_k(m)) = 0$. We now derive the variance of $L_k(m)$:

$$\begin{aligned}
\mathrm{Var}(L_k(m)) &= \mathrm{Cov}\left[L_k(m), L_k(m)\right] \\
&= \frac{12}{m^2}\mathrm{Cov}\left[\sum_{i=1}^{m} \mathcal{B}_k\left(\frac{i}{m}\right), \sum_{j=1}^{m} \mathcal{B}_k\left(\frac{j}{m}\right)\right] \\
&= \frac{12}{m^2}\sum_{i=1}^{m}\sum_{j=1}^{m} \mathrm{Cov}\left[\mathcal{B}_k\left(\frac{i}{m}\right), \mathcal{B}_k\left(\frac{j}{m}\right)\right] \\
&= \frac{12}{m^2}\sum_{i=1}^{m} \mathrm{Var}\left[\mathcal{B}_k\left(\frac{i}{m}\right)\right] + \frac{24}{m^2}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \mathrm{Cov}\left[\mathcal{B}_k\left(\frac{i}{m}\right), \mathcal{B}_k\left(\frac{j}{m}\right)\right] \\
&= \frac{12}{m^2}\sum_{i=1}^{m} \frac{i}{m}\left(1 - \frac{i}{m}\right) + \frac{24}{m^2}\sum_{i=1}^{m-1}\sum_{j=i+1}^{m} \left(\frac{i}{m} - \frac{ij}{m^2}\right).
\end{aligned}$$

Using $\sum_{i=1}^{m} i = m(m+1)/2$, $\sum_{i=1}^{m} i^2 = m(m+1)(2m+1)/6$, $\sum_{i=1}^{m} i^3 = m^2(m+1)^2/4$, and some algebra we get

$$\mathrm{Var}(L_k(m)) = \frac{(m+1)(m^2-m)}{m^3} = \frac{m^2-1}{m^2}. \tag{2.2.3}$$

Equation (2.2.2) implies that the characteristic function of $L_k(m)$ converges to the characteristic function of N_k as $m \to \infty$ (see Grimmett and Stirzaker [29, p. 172]). Since $L_k(m)$ is normal with mean 0 and variance given by Equation (2.2.3), its characteristic function is given by

$$\phi_m(t) = \mathrm{E}(\exp(itL_k(m))) = \exp\left(-\frac{t^2}{2}\left(\frac{m^2-1}{m^2}\right)\right),$$

where $i \equiv \sqrt{-1}$. Since the exponential function is continuous, we have

$$\phi_m(t) \to \exp(-t^2/2) \quad \text{as } m \to \infty.$$

We conclude that $N_k \sim \mathrm{Nor}(0,1)$ since $\exp(-t^2/2)$ is the characteristic function of the standard normal distribution. $\square$

Lemma 2.2.8 *For fixed k, $(N_1, N_2, \ldots, N_k)$ has a non-singular multivariate normal distribution.*

Proof: First, we show that every linear combination $\sum_{j=1}^{k} a_j N_j$ has normal distribution, and hence, $\mathbf{N} \equiv (N_1, \ldots, N_k)$ has multivariate normal distribution by virtue of Theorem

2.6.2 of Anderson [4]. Indeed,

$$\sum_{j=1}^{k} a_j N_j = \sum_{j=1}^{k} a_j \int_0^1 \mathcal{B}_j(t)dt = \int_0^1 \left[\sum_{j=1}^{k} a_j \mathcal{B}_j(t)\right] dt.$$

Further, by the representation (2.1.4) of each $\mathcal{B}_j(t)$,

$$\begin{aligned}\mathcal{Z}(t) \equiv \sum_{j=1}^{k} a_j \mathcal{B}_j(t) &= \sum_{j=1}^{k} \sum_{i=1}^{2^{j-1}} a_j \left[\mathcal{W}\left(\frac{i-1}{2^{j-1}} + \frac{t}{2^j}\right) - \mathcal{W}\left(\frac{i}{2^{j-1}} - \frac{t}{2^j}\right)\right] \\ &\quad + \left(\sum_{j=1}^{k} a_j\right)(1-t)\mathcal{W}(1).\end{aligned}$$

Now, let $c_1, \ldots, c_m$ be real constants and $0 \le t_1 < \cdots < t_n \le 1$. Then,

$$\begin{aligned}\sum_{l=1}^{n} c_l \mathcal{Z}(t_l) &= \sum_{l=1}^{n} c_l \sum_{j=1}^{k} a_j \sum_{i=1}^{2^{j-1}} \left[\mathcal{W}\left(\frac{i-1}{2^{j-1}} + \frac{t_l}{2^j}\right) - \mathcal{W}\left(\frac{i}{2^{j-1}} - \frac{t_l}{2^j}\right)\right] \\ &\quad + \sum_{l=1}^{n} c_l \left(\sum_{j=1}^{k} a_j\right)(1-t)\mathcal{W}(1).\end{aligned}$$

Let $\mathcal{T}$ be the set of all times of the form $\frac{i-1}{2^{j-1}} + \frac{t_l}{2^j}$ or $\frac{i}{2^{j-1}} - \frac{t_l}{2^j}$, for some $l = 1, \ldots, n$, $j = 1, \ldots, k$, and $i = 1, \ldots, 2^{j-1}$. Let $\{\tau_1, \ldots, \tau_N\}$ be an increasing ordering of $\mathcal{T} \cup \{1\}$. Clearly, we can write $\sum_{l=1}^{n} c_l \mathcal{Z}(t_l)$ as $\sum_{m=1}^{N} d_m \mathcal{W}(\tau_m)$, for some real constants $d_1, \ldots, d_N$. Since $\mathcal{W}$ is a Gaussian process, the latter summation is Gaussian and thus, $\mathcal{Z}$ is a Gaussian process. Notice also that $\mathcal{Z}$ has continuous paths because $\mathcal{W}$ has continuous paths. Then, as in the proof of Lemma 2.2.7, $\int_0^1 \mathcal{Z}(t)\,dt$ is Gaussian.

To prove that $\mathbf{N} = (N_1, \ldots, N_k)$ has nonsingular multivariate normal distribution, we have to show that the variance-covariance matrix $\mathbf{\Sigma_N}$ is positive definite. This follows inmediately from Lemmas 2.2.5 and 2.2.6 since

$$\mathbf{a}\mathbf{\Sigma_N}\mathbf{a}^T = \mathrm{Var}\left(\sum_{j=1}^{k} a_j N_j\right) = \sum_{j=1}^{k} a_j^2 > 0,$$

for all $\mathbf{a} = (a_1, \ldots, a_k)$ in $\mathbb{R}^k - \{\mathbf{0}\}$. $\square$

Proof of Theorem 2.2.2: Lemma 2.2.6 implies that $\mathrm{Cov}(N_k, N_j) = 0$ for every $k \neq j$. Now, since by Lemma 2.2.8 $(N_1, N_2, \ldots, N_k)$ has a multivariate normal distribution, we can conclude that the random variables $N_1, N_2, \ldots$ are i.i.d. Nor$(0, 1)$. $\square$

2.3 *Definition and Properties of Folded Standardized Time Series*

In Chapters III and IV we will introduce two new estimators for the variance parameter of a stationary stochastic process: the folded version of the weighted area estimator, and the folded version of the weighted Cramér-von Mises estimator. In both cases, we will compute the asymptotic expected values and variances. In this section, we will form an appropriate folded STS that converges to the corresponding folded Brownian bridge; this folded STS will be used to define the estimators in the upcoming chapters.

The following definition is similar to the definition of the level-1 Brownian bridge (Definition 2.1.2).

Definition 2.3.1

$$T_n^{(1)}(t) \equiv T_n^{(0)}\left(\frac{t}{2}\right) - T_n^{(0)}\left(1-\frac{t}{2}\right) \quad \textit{for } 0 \le t \le 1.$$

where $T_n^{(0)}(t) \equiv T_n(t)$ *is defined in Equation (1.1.2).*

The next lemma will also be useful in the upcoming chapters.

Lemma 2.3.2 *For any integer* j*, with* $j \le n$*,*

$$T_n^{(1)}\left(\frac{j}{n}\right) = \frac{1}{\sigma\sqrt{n}}\left\{(j-n)\,\overline{X}_n - Z_{\lfloor \frac{j}{2} \rfloor} + Z_{\lfloor n-\frac{j}{2} \rfloor}\right\},$$

where $Z_k \equiv \sum_{i=1}^{k} X_i$*, for* $k = 1, \ldots, n$*, and* $Z_0 \equiv 0$*.*

Proof:

$$\begin{aligned}
T_n^{(1)}(t) &= T_n^{(0)}\left(\frac{t}{2}\right) - T_n^{(0)}\left(1-\frac{t}{2}\right) \\
&= \frac{\lfloor \frac{nt}{2} \rfloor \left(\bar{X}_n - \bar{X}_{\lfloor \frac{nt}{2} \rfloor}\right)}{\sigma\sqrt{n}} - \frac{\lfloor n(1-\frac{t}{2}) \rfloor \left(\bar{X}_n - \bar{X}_{\lfloor n(1-\frac{t}{2}) \rfloor}\right)}{\sigma\sqrt{n}} \\
&= \frac{1}{\sigma\sqrt{n}}\left\{\left(\left\lfloor \frac{nt}{2} \right\rfloor - \left\lfloor n\left(1-\frac{t}{2}\right)\right\rfloor\right)\overline{X}_n - Z_{\lfloor \frac{nt}{2} \rfloor} + Z_{\lfloor n(1-\frac{t}{2}) \rfloor}\right\}.
\end{aligned}$$

The proof follows from the substitution $t = j/n$ and the fact that $\lfloor j/2 \rfloor - \lfloor n - j/2 \rfloor = j - n$.

□

Similarly to how we defined the level-k Brownian bridge, we give the following definition.

Definition 2.3.3 *The level-k STS is defined recursively as follows:*

$$T_n^{(k)}(t) \equiv T_n^{(k-1)}\left(\frac{t}{2}\right) - T_n^{(k-1)}\left(1 - \frac{t}{2}\right) \quad \textit{for } 0 \leq t \leq 1.$$

The following lemma gives an equation relating the level-k STS with the original (level-0) STS, and its proof parallels the proof of Lemma 2.1.5.

Lemma 2.3.4 *For $k \geq 1$,*

$$T_n^{(k)}(t) = \sum_{i=1}^{2^{k-1}} \left[T_n^{(0)}\left(\frac{i-1}{2^{k-1}} + \frac{t}{2^k}\right) - T_n^{(0)}\left(\frac{i}{2^{k-1}} - \frac{t}{2^k}\right)\right]. \tag{2.3.1}$$

Proof: We will carry out the proof by induction. First, we express the level-2 STS in terms of the level-0 STS:

$$\begin{aligned}
T_n^{(2)}(t) &= T_n^{(1)}\left(\frac{t}{2}\right) - T_n^{(1)}\left(1 - \frac{t}{2}\right) \\
&= T_n^{(0)}\left(\frac{t}{4}\right) - T_n^{(0)}\left(\frac{1}{2} - \frac{t}{4}\right) + T_n^{(0)}\left(\frac{1}{2} + \frac{t}{4}\right) - T_n^{(0)}\left(1 - \frac{t}{4}\right) \\
&= \sum_{i=1}^{2} \left[T_n^{(0)}\left(\frac{i-1}{2} + \frac{t}{4}\right) - T_n^{(0)}\left(\frac{i}{2} - \frac{t}{4}\right)\right],
\end{aligned}$$

thanks to Definition 2.3.1. Therefore, Equation (2.3.1) holds for $k = 2$.

Now, let us assume by the inductive hypothesis that, for all $0 \leq t \leq 1$,

$$T_n^{(k-1)}(t) = \sum_{i=1}^{2^{k-2}} \left[T_n^{(0)}\left(\frac{i-1}{2^{k-2}} + \frac{t}{2^{k-1}}\right) - T_n^{(0)}\left(\frac{i}{2^{k-2}} - \frac{t}{2^{k-1}}\right)\right].$$

Then, using Definition 2.3.3, we have

$$\begin{aligned}
T_n^{(k)}(t) &= T_n^{(k-1)}\left(\frac{t}{2}\right) - T_n^{(k-1)}\left(1 - \frac{t}{2}\right) \\
&= \sum_{i=1}^{2^{k-2}} \left[T_n^{(0)}\left(\frac{i-1}{2^{k-2}} + \frac{t}{2^k}\right) - T_n^{(0)}\left(\frac{i}{2^{k-2}} - \frac{t}{2^k}\right)\right] \\
&\quad - \sum_{i=1}^{2^{k-2}} \left[T_n^{(0)}\left(\frac{2i-1}{2^{k-1}} - \frac{t}{2^k}\right) - T_n^{(0)}\left(\frac{2i-1}{2^{k-1}} + \frac{t}{2^k}\right)\right] \\
&\qquad \text{(by the inductive hypothesis)} \\
&= \sum_{i=1}^{2^{k-2}} \left[T_n^{(0)}\left(\frac{2(i-1)}{2^{k-1}} + \frac{t}{2^k}\right) - T_n^{(0)}\left(\frac{2i}{2^{k-1}} - \frac{t}{2^k}\right)\right] \\
&\quad - \sum_{i=1}^{2^{k-2}} \left[T_n^{(0)}\left(\frac{2i-1}{2^{k-1}} - \frac{t}{2^k}\right) - T_n^{(0)}\left(\frac{2i-1}{2^{k-1}} + \frac{t}{2^k}\right)\right]
\end{aligned}$$

$$= \sum_{i=1}^{2^{k-2}} \left[T_n^{(0)} \left(\frac{2(i-1)}{2^{k-1}} + \frac{t}{2^k} \right) - T_n^{(0)} \left(\frac{2i-1}{2^{k-1}} - \frac{t}{2^k} \right) \right]$$
$$+ \sum_{i=1}^{2^{k-2}} \left[T_n^{(0)} \left(\frac{2i-1}{2^{k-1}} + \frac{t}{2^k} \right) - T_n^{(0)} \left(\frac{2i}{2^{k-1}} - \frac{t}{2^k} \right) \right]$$

(rearranging the sums)

$$= \sum_{j \text{ odd}} \left[T_n^{(0)} \left(\frac{j-1}{2^{k-1}} + \frac{t}{2^k} \right) - T_n^{(0)} \left(\frac{j}{2^{k-1}} - \frac{t}{2^k} \right) \right]$$
$$+ \sum_{j \text{ even}} \left[T_n^{(0)} \left(\frac{j-1}{2^{k-1}} + \frac{t}{2^k} \right) - T_n^{(0)} \left(\frac{j}{2^{k-1}} - \frac{t}{2^k} \right) \right]$$

(taking $j = 2i - 1$ $(1 \le j \le 2^{k-1} - 1)$ in the first sum

and $j = 2i$ $(2 \le j \le 2^{k-1})$ in the second sum)

$$= \sum_{j=1}^{2^{k-1}} \left[T_n^{(0)} \left(\frac{j-1}{2^{k-1}} + \frac{t}{2^k} \right) - T_n^{(0)} \left(\frac{j}{2^{k-1}} - \frac{t}{2^k} \right) \right] \quad \text{for } 0 \le t \le 1. \quad \square$$

The next lemma relates the level-k STS with the original underlying process.

Lemma 2.3.5 *For any integer j, with $j \le n$,*

$$T_n^{(k)}\left(\frac{j}{n}\right) = \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{2^{k-1}} \left(\left\lfloor n \left(\frac{i-1}{2^{k-1}} + \frac{j}{n2^k} \right) \right\rfloor - \left\lfloor n \left(\frac{i}{2^{k-1}} - \frac{j}{n2^k} \right) \right\rfloor \right) \overline{X}_n$$
$$+ \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{2^{k-1}} \left\{ Z_{\left\lfloor n\left(\frac{i}{2^{k-1}} - \frac{j}{n2^k}\right)\right\rfloor} - Z_{\left\lfloor n\left(\frac{i-1}{2^{k-1}} + \frac{j}{n2^k}\right)\right\rfloor} \right\}.$$

Proof: By Lemma 2.3.4,

$$T_n^{(k)}(t) = \sum_{i=1}^{2^{k-1}} \left[T_n^{(0)} \left(\frac{i-1}{2^{k-1}} + \frac{t}{2^k} \right) - T_n^{(0)} \left(\frac{i}{2^{k-1}} - \frac{t}{2^k} \right) \right]$$
$$= \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{2^{k-1}} \left\lfloor n \left(\frac{i-1}{2^{k-1}} + \frac{t}{2^k} \right) \right\rfloor \left(\overline{X}_n - \overline{X}_{\left\lfloor n\left(\frac{i-1}{2^{k-1}} + \frac{t}{2^k}\right)\right\rfloor} \right)$$
$$- \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{2^{k-1}} \left\lfloor n \left(\frac{i}{2^{k-1}} - \frac{t}{2^k} \right) \right\rfloor \left(\overline{X}_n - \overline{X}_{\left\lfloor n\left(\frac{i}{2^{k-1}} - \frac{t}{2^k}\right)\right\rfloor} \right)$$

(by Definition 1.1.2)

$$= \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{2^{k-1}} \left(\left\lfloor n \left(\frac{i-1}{2^{k-1}} + \frac{t}{2^k} \right) \right\rfloor - \left\lfloor n \left(\frac{i}{2^{k-1}} - \frac{t}{2^k} \right) \right\rfloor \right) \overline{X}_n$$
$$+ \frac{1}{\sigma\sqrt{n}} \sum_{i=1}^{2^{k-1}} \left\{ Z_{\left\lfloor n\left(\frac{i}{2^{k-1}} - \frac{t}{2^k}\right)\right\rfloor} - Z_{\left\lfloor n\left(\frac{i-1}{2^{k-1}} + \frac{t}{2^k}\right)\right\rfloor} \right\}$$

(regrouping the terms).

The proof follows from the substitution $t = j/n$. $\square$

CHAPTER III

FOLDED WEIGHTED AREA VARIANCE ESTIMATOR FOR SIMULATIONS

In this chapter, we introduce the folded version of Schruben's weighted area estimator for σ^2. Its asymptotic properties and distribution are given in Section 3.2. In Section 3.3, we identify some weight functions that yield first-order unbiased estimators. Finally, in Section 3.4 we compute the covariance between the two first levels (level-0 and level-1) of the weighted area estimator, and provide weight functions for which this covariance is zero.

3.1 Definitions

Definition 3.1.1 *For each $i \geq 1$, the level-i folded weighted STS area estimator for σ^2 is defined by*

$$A_i(w,n) \equiv N_i^2(w,n) \tag{3.1.1}$$

where

$$N_i(w,n) \equiv \frac{1}{n}\sum_{k=1}^{n} w\left(\frac{k}{n}\right)\sigma T_n^{(i)}\left(\frac{k}{n}\right),$$

and $T_n^{(i)}(\cdot)$ is given by Definition 2.3.3, and $w(\cdot)$ is a weight function satisfying Assumption A.5.

Definition 3.1.2 *For each $i \geq 1$, the square of the weighted area under the level-i Brownian bridge, $\mathcal{B}_i(t)$, is defined by*

$$A_i(w) \equiv N_i^2(w) \tag{3.1.2}$$

where

$$N_i(w) \equiv \int_0^1 w\left(t\right)\sigma\mathcal{B}_i(t)dt \tag{3.1.3}$$

and $w(\cdot)$ is a weight function satisfying Assumption A.5.

3.2 Asymptotic Results

In this section we present two important results concerning the estimator $A_i(w,n)$. These are Theorems 3.2.1 and 3.2.3. Theorem 3.2.1 gives us the distribution of $A_i(w)$ for each $i \geq 1$ as well as the weak convergence of $A_i(w,n)$ to that distribution. This result will be useful, e.g., when constructing confidence intervals for the mean μ of the underlying stochastic process using $A_i(w,n)$ as the estimator for σ^2. Since the technical details of the proof of Theorem 3.2.1 are rather involved, we will present them in Section A.2 of Appendix A.

Theorem 3.2.1 *If Assumptions A hold, then*

$$A_i(w,n) \xrightarrow{\mathcal{D}} A_i(w) \sim \sigma^2 \chi_1^2 \quad \text{as } n \to \infty \text{ for all } i \geq 1. \tag{3.2.1}$$

Further $\sqrt{n}\big(\overline{X}_n - \mu_X\big)$ *and* $A_i(w,n)$ *are independent as* $n \to \infty$.

Remark 3.2.2 Note that Theorem 3.2.1 only requires the weight function $w(\cdot)$ to be continuous on $[0,1]$. The existence of derivatives of $w(\cdot)$ is not necessary to establish the asymptotic distribution of the folded weighted area estimator. $\triangleleft$

Theorem 3.2.3 gives asymptotic expressions for the expected value and variance of the level-1 folded weighted STS area estimator.

Theorem 3.2.3 *Suppose* $\{X_i, i \geq 1\}$ *is a stochastic process satisfying Assumptions A. Also assume that then sequence* $\{A_1^2(w,n), n \geq 1\}$ *is uniformly integrable. Let*

$$\begin{aligned} W_n^* = W^2 - \overline{W}^2 &+ \frac{w(0)}{n}(2\overline{W} - W) \\ &- 2\int_0^{1/2} u\check{w}_n(u)\check{W}_n(u)\,du - 2\int_0^{1/2} u\widehat{w}_n(u)\widehat{W}_n(u)\,du \\ &+ \frac{\check{w}_n(0)}{n}\int_0^{1/2} u\check{w}_n(u)\,du \end{aligned} \tag{3.2.2}$$

where $W(\cdot)$ and $\overline{W}$ are defined in Theorem 1.2.5, and for $n \geq 1$,

$$\widehat{w}_n(t) \equiv w(2t) + w\left(2t + 1/n\right) \quad \text{for } 0 \leq t \leq 1/2, \tag{3.2.3}$$

$$\widehat{W}_n(t) \equiv \int_0^t \widehat{w}_n(s)\, ds \quad \text{for } 0 \leq t \leq 1/2, \tag{3.2.4}$$

$$\check{w}_n(t) \equiv w(2t) + w\left(2t - 1/n\right) \quad \text{for } 0 \leq t \leq 1/2, \quad \text{and} \tag{3.2.5}$$

$$\check{W}_n(t) \equiv \int_0^t \check{w}_n(s)\, ds \quad \text{for } 0 \leq t \leq 1/2. \tag{3.2.6}$$

Then

$$\mathrm{E}(A_1(w,n)) = W_n^* \sigma^2 + \frac{\overline{W}^2 \gamma}{n} + \mathrm{o}(1/n) \tag{3.2.7}$$

and

$$\mathrm{Var}(A_1(w,n)) \to \mathrm{Var}(A_1(w)) = \mathrm{Var}(\sigma^2 \chi_1^2) = 2\sigma^4 \quad \text{as } n \to \infty. \tag{3.2.8}$$

Definition 3.2.4 *Let $W_{D,n}^*$ be the exact, but still unknown coefficient, of σ^2 in $\mathrm{E}(A_1(w,n))$. We will get an explicit expression for $W_{D,n}^*$ during the proof of Theorem 3.2.3 in Appendix B via Equation (B.3.1). Since the expression requires rather involved notation that we have not established yet, we will not display it here.*

Remark 3.2.5 Note that $A_1(w,n)/W_n^*$ is a slightly better estimator than $A_1(w,n)$ due to the fact that $\mathrm{E}(A_1(w,n)/W_n^*) = \sigma^2 + \mathrm{O}(1/n)$ in Equation 3.2.7. Actually, we will see that W_n^* (the approximate coefficient of σ^2 in $\mathrm{E}(A_1(w,n))$) and $W_{D,n}^*$ the exact coefficient of σ^2 in $\mathrm{E}(A_1(w,n))$ are both so close to one for the weights we consider herein that using them as correction factors may not be necessary. For more details turn to Section 5.6, where the correction factors are computed for a specific weight function (see Table 7).

Remark 3.2.6 Notice that $\mathrm{Var}(A_1(w))$ does not depend on the weight function $w(\cdot)$, so we will not be able to choose weights to reduce the variance of our estimator—in contrast with the folded weighted CvM estimator we introduce in Chapter IV. $\triangleleft$

Remark 3.2.7 Due to the length and technical details involved in the proof of Theorem 3.2.3, we defer it to Section B.3 in Appendix B. However, we will include next a description of the main idea of the proof, the intermediate results required to complete the proof, as well as the methods used to establish those results.

- We develop an expression for $T_n^{(1)}(j/n)$ for all $j = 1, 2, \ldots, n$ in terms of the underlying stochastic process $\{X_i, i \geq 1\}$. This equation is given by Lemma 2.3.2, and it allows us to use results from Goldsman and Meketon [25] for the variance and covariances involving partial sums of the observations of a stationary stochastic process satisfying Assumptions A.1–A.4.

- We note that $\mathrm{E}(N_1^2(w,n)) = \mathrm{Var}(N_1(w,n)) = \mathrm{Cov}(N_1(w,n), N_1(w,n))$ as a result of the fact that $\mathrm{E}(N_1(w,n)) = 0$.

- Once we have expressed $\mathrm{E}(N_1^2(w,n))$ in terms the variances and covariances of the partial sums of the original observations, we use Equations (B.1.1)–(B.1.5) from Appendix B. Most of these equations were taken from Goldsman and Meketon [25], though some were derived by us.

- We proceed next to give notation with the objective of writing the summations involved in a more compact way, so that we can represent certain discrete approximations to integral expressions. A lot of algebraic artillery was needed in this segment of the proof. Notation details are given in Section B.1, and the derivation of the compact summations is given in Lemmas B.2.1–B.2.6.

- To substitute some of the terms in $\mathrm{E}(A_1(w,n))$ by $\mathrm{o}(1/n)$, many bounding tricks were needed in tandem with a great deal of tedious algebra. Also, Assumptions A.4 and A.5 are required in this part of the proof. The tolerant reader can go through the details in Lemmas B.2.7–B.2.14.

- Finally, we use the Trapezoid Rule for integrals to substitute our summations by their corresponding integrals. Also, in this part we use the continuity of $w(\cdot)$ given by Assumption A.5. Details are given in Lemmas B.2.16–B.2.22. $\triangleleft$

3.3 Some Interesting Weight Functions

Example 3.3.1 For $w_0(t) \equiv \sqrt{12}$ for all $0 \le t \le 1$, Theorem 3.2.3 implies that

$$\mathrm{E}(N_1^2(w_0, n)) = \left(1 + \frac{6}{n}\right)\sigma^2 + \frac{3\gamma}{n} + \mathrm{o}(1/n),$$

which is similar to the result found for the unfolded case (see Example 1.2.6). ◁

Example 3.3.2 If given $k \ge 1$, we take

$$w_{\sin,k}(t) = \frac{8\sqrt{3}\pi^2 k^2 t \sin(2k\pi t)}{\sqrt{15 + 8k^2\pi^2}},$$

Theorem 3.2.3 implies

$$\mathrm{E}(N_1^2(w_{\sin,k}, n)) = H(k, n)\sigma^2 + \mathrm{o}(1/n),$$

where

$$\begin{aligned}(15 + 8k^2\pi^2)n^2 H(k, n) \equiv \cos^4(k\pi/n)&(48k^2\pi^2(1 - n^2) - 18n^2)\\ &- \cos^3(k\pi/n)\sin(k\pi/n)(72kn\pi)\\ &+ \cos^2(k\pi/n)(8k^2\pi^2(n^2 - 6) + 33n^2)\\ &+ \cos(k\pi/n)\sin(k\pi/n)(54kn\pi)\\ &+ (6k^2\pi^2(8n^2 + 3)).\end{aligned}$$

Notice that $H(k, n) \to 1$ as $n \to \infty$ for each $k \ge 1$. Therefore, the estimator $N_1^2(w_{\sin,k}, n)/H(k, n)$ is first-order unbiased for each $k \ge 1$. ◁

3.4 Covariance Between Two Folded Estimators

Even though the limiting variance of the weighted area estimator cannot be reduced by changing the weight function, the covariance between the square of the weighted area under the level-0 Brownian bridge and the square of the weighted area under the level-1 Brownian bridge can. As a result, we can examine the variance of linear combinations of different levels of the weighted area estimators, with the ultimate goal of producing new estimators with comparatively smaller variance than before while maintaining the same low bias. Theorem 3.4.1 computes such covariances with the final objective of studying the asymptotic behavior of the variance of the new estimators described above. In particular, we are interested in the average of the first two levels of the weighted area estimator for σ^2. At the end of this section, we will compute the variances of the these combined estimators for the weight functions from Examples 1.2.6, 1.2.8, and 3.3.2.

Theorem 3.4.1 *Let $f_i(\cdot)$ be a function satisfying Assumption A.5 for $i = 0, 1$. For $i = 0, 1$, let $F_i(s) \equiv \int_0^s f_i(u)\,du$ for $0 \le s \le 1$, and $\overline{F}_i(t) \equiv \int_0^t F_i(s)\,ds$ for $0 \le t \le 1$. Then, under Assumptions A.1–A.4,*

$$\mathrm{Cov}(A_0(f_0), A_1(f_1)) = 2\sigma^4 \left[\int_0^1 f_1(s)\left\{\overline{F}_0\left(1 - \frac{s}{2}\right) - \overline{F}_0\left(\frac{s}{2}\right)\right\} ds - \overline{F}_0(1)\overline{F}_1(1)\right]^2.$$

Proof: First, observe that

$$\begin{aligned}
\mathrm{Cov}(A_0(f_0), A_1(f_1)) &= \mathrm{Cov}\left(\left[\int_0^1 f_0(t)\sigma\mathcal{B}_0\,(t)\;dt\right]^2, \left[\int_0^1 f_1(s)\sigma\mathcal{B}_1\,(s)\;ds\right]^2\right) \\
&= 2\mathrm{Cov}^2\left(\int_0^1 f_0(t)\sigma\mathcal{B}_0\,(t)\;dt, \int_0^1 f_1(s)\sigma\mathcal{B}_1\,(s)\;ds\right) \\
&\qquad \text{(since } \mathcal{B}_k\,(\cdot) \text{ is a Gaussian process with } \mathrm{E}(\mathcal{B}_k\,(t)) = 0 \\
&\qquad \text{for every } k \ge 0 \text{ and } 0 \le t \le 1\text{; see Patel and Read [36])} \\
&= 2\sigma^4\left[\int_0^1\int_0^1 f_0(t)f_1(s)\mathrm{Cov}(\mathcal{B}_0\,(t)\,, \mathcal{B}_1\,(s))\,dt\,ds\right]^2. \qquad (3.4.1) \\
&\qquad \text{(by Fubini's Theorem and the continuity} \\
&\qquad \text{of } f_0(t)f_1(s)\mathcal{B}_0\,(t)\,\mathcal{B}_1\,(s) \text{ on } [0,1]^2\text{).}
\end{aligned}$$

Now, we have

$$\begin{aligned}
\mathrm{Cov}(\mathcal{B}_0\,(t)\,, \mathcal{B}_1\,(s)) &= \mathrm{Cov}(\mathcal{B}_0\,(t)\,, \mathcal{B}_0\,(s/2) - \mathcal{B}_0\,(1 - s/2)) \\
&= \begin{cases} t(1-s) & \text{if } t < \frac{s}{2} \\ s(\frac{1}{2} - t) & \text{if } \frac{s}{2} < t < 1 - \frac{s}{2} \\ (s-1)(1-t) & \text{if } 1 - \frac{s}{2} < t. \end{cases} \qquad (3.4.2)
\end{aligned}$$

Plugging (3.4.2) into (3.4.1), we obtain

$$
\begin{aligned}
&\operatorname{Cov}(A_0(f_0), A_1(f_1)) \\
&= 2\sigma^4 \Bigg[\int_0^1 f_1(s) \Bigg\{ (1-s) \int_0^{\frac{s}{2}} f_0(t) t\, dt \\
&\qquad + s \int_{\frac{s}{2}}^{1-\frac{s}{2}} f_0(t) \left(\frac{1}{2} - t \right) dt + (s-1) \int_{1-\frac{s}{2}}^{1} f_0(t)(1-t)\, dt \Bigg\} ds \Bigg]^2 .
\end{aligned} \tag{3.4.3}
$$

Let $\mathcal{J}$ denote the quantity in braces in Equation (3.4.3). Then

$$
\begin{aligned}
\mathcal{J} &= -s \int_0^1 f_0(t) t\, dt + \int_0^{\frac{s}{2}} f_0(t) t\, dt + \int_{1-\frac{s}{2}}^{1} f_0(t) t\, dt \\
&\qquad + \frac{s}{2} \int_{\frac{s}{2}}^{1-\frac{s}{2}} f_0(t)\, dt + (s-1) \int_{1-\frac{s}{2}}^{1} f_0(t)\, dt \\
&= -s \int_0^1 f_0(t) t\, dt + \int_0^1 f_0(t) t\, dt - \int_{\frac{s}{2}}^{1-\frac{s}{2}} f_0(t) t\, dt \\
&\qquad + \frac{s}{2} \left(F_0\!\left(1 - \frac{s}{2}\right) - F_0\!\left(\frac{s}{2}\right) \right) + (s-1) \left(F_0(1) - F_0\!\left(1 - \frac{s}{2}\right) \right) \\
&= (1-s) \int_0^1 f_0(t) t\, dt - \int_{\frac{s}{2}}^{1-\frac{s}{2}} f_0(t) t\, dt \\
&\qquad + \frac{s}{2} \left(F_0\!\left(1 - \frac{s}{2}\right) - F_0\!\left(\frac{s}{2}\right) \right) + (s-1) \left(F_0(1) - F_0\!\left(1 - \frac{s}{2}\right) \right).
\end{aligned} \tag{3.4.4}
$$

Using integration by parts, we get

$$
\int_a^b f_0(t) t\, dt = F_0(t) t \Big|_a^b - \int_a^b F_0(t)\, dt = F_0(b) b - F_0(a) a - \overline{F}_0(b) + \overline{F}_0(a). \tag{3.4.5}
$$

Plugging (3.4.5) into (3.4.4), we obtain (after a little algebra)

$$
\begin{aligned}
\mathcal{J} &= (1-s)(F_0(1) - \overline{F}_0(1)) \\
&\qquad - \left[F_0\!\left(1 - \frac{s}{2}\right)\left(1 - \frac{s}{2}\right) - F_0\!\left(\frac{s}{2}\right)\left(\frac{s}{2}\right) - \overline{F}_0\!\left(1 - \frac{s}{2}\right) + \overline{F}_0\!\left(\frac{s}{2}\right) \right] \\
&\qquad + \frac{s}{2} \left[F_0\!\left(1 - \frac{s}{2}\right) - F_0\!\left(\frac{s}{2}\right) \right] + (s-1) \left[F_0(1) - F_0\!\left(1 - \frac{s}{2}\right) \right] \\
&= (s-1)\overline{F}_0(1) + \overline{F}_0\!\left(1 - \frac{s}{2}\right) - \overline{F}_0\!\left(\frac{s}{2}\right).
\end{aligned} \tag{3.4.6}
$$

Plugging (3.4.6) into (3.4.3), and employing integration by parts again, we obtain

$$
\begin{aligned}
&\mathrm{Cov}(A_0(f_0), A_1(f_1)) \\
&= 2\sigma^4 \left[\int_0^1 f_1(s) \left\{ (s-1)\overline{F}_0(1) + \overline{F}_0\left(1 - \frac{s}{2}\right) - \overline{F}_0\left(\frac{s}{2}\right) \right\} ds \right]^2 \\
&= 2\sigma^4 \left[-\overline{F}_0(1)\overline{F}_1(1) + \int_0^1 f_1(s) \left\{ \overline{F}_0\left(1 - \frac{s}{2}\right) - \overline{F}_0\left(\frac{s}{2}\right) \right\} ds \right]^2. \quad \square
\end{aligned} \tag{3.4.7}
$$

Remark 3.4.2 If we take $f_0(t) = f_1(t) = w_0(t) = \sqrt{12}$, then

$$
\int_0^1 f_1(s) \left\{ \overline{F}_0\left(1 - \frac{s}{2}\right) - \overline{F}_0\left(\frac{s}{2}\right) \right\} ds - \overline{F}_0(1)\overline{F}_1(1) = 0. \tag{3.4.8}
$$

Moreover, (3.4.8) is zero as well if we take $f_0(t) = w_{\cos,k}(t)$ for any $k \geq 1$ and any choice of $f_1(t)$. This is an extremely pleasant surprise since it will allow us to have a variance reduction of 50% with respect to the original weighted area estimators for σ^2. See Chapter V for additional details. $\triangleleft$

CHAPTER IV

FOLDED WEIGHTED CRAMÉR-VON MISES VARIANCE ESTIMATOR FOR SIMULATIONS

In this chapter, we start with the definition of the folded version of the weighted CvM estimator for the variance parameter σ^2. We derive its asymptotic properties in Section 4.2. In Section 4.3, we give several examples of families of weight functions yielding first-order unbiased estimators. In this case, in contrast with the folded weighted area variance estimator, we will be able to choose the member of any such family that minimizes the limiting variance of the folded weighted Cramér-von Mises variance estimator.

4.1 Definitions

Definition 4.1.1 *For $i \geq 0$, the level-i folded weighted CvM estimator for σ^2 is defined by*

$$C_i(g,n) \equiv \frac{1}{n}\sum_{j=1}^{n} g\left(\frac{j}{n}\right)\left[\sigma T_n^{(i)}\left(\frac{j}{n}\right)\right]^2,$$

where $T_n^{(i)}(\cdot)$ is given by Definition 2.3.3, and $g(\cdot)$ is a weight function satisfying Assumption A.6.

Definition 4.1.2 *For $i \geq 0$, the weighted area under the square of the level-i Brownian bridge, $\mathcal{B}_i(t)$ is defined by*

$$C_i(g) \equiv \int_0^1 g\left(t\right)\left(\sigma \mathcal{B}_i(t)\right)^2\, dt$$

Theorem 4.1.3 *Under Assumptions A, we have that*

$$C_i(g,n) \xrightarrow{\mathcal{D}} C_i(g) \quad \text{as } n \to \infty \text{ for all } i \geq 1. \tag{4.1.1}$$

Remark 4.1.4 Since the details of the proof of Theorem 4.1.3 are rather involved, we defer it to Appendix A. $\triangleleft$

4.2 Asymptotic Results

Theorem 4.2.1 gives asymptotic results on the expected value and limiting variance of the level-1 folded weighted CvM estimator. Its detailed proof can be found in Section B.5 of Appendix B.

Theorem 4.2.1 *Suppose $\{X_i, i \geq 1\}$ is a stochastic process satisfying Assumptions A.1–A.4, and let $g(\cdot)$ be a weight function satisfying Assumption A.6 Further, suppose that the sequence $\{C_1^2(g,n), n \geq 1\}$ is uniformly integrable, and let*

$$G^* \equiv \int_0^1 \int_0^t g(s)\, ds dt - \int_0^1 (1-t)^2 g(t)\, dt = 1$$

and

$$I \equiv \int_0^1 (t^2 - 2t + 2) g(t)\ dt.$$

Then

$$\mathrm{E}(C_1(g,n)) = G^* \sigma^2 + \frac{I\gamma}{n} + \mathrm{o}(1/n)$$

and

$$\mathrm{Var}(C_1(g,n)) \to \mathrm{Var}(C_1(g)) = 4\sigma^4 \int_0^1 g(t)(1-t)^2 \int_0^t g(s) s^2\ ds\ dt \quad \text{as } n \to \infty.$$

Remark 4.2.2 Note that we can choose $g(\cdot)$ such that $G^* \equiv 1$ and $I \equiv 0$ to yield first-order unbiased estimators. $\triangleleft$

Remark 4.2.3 The techniques used in the proof of Theorem 4.2.1 parallel the ones used in the proof of Theorem 3.2.3. Assumption A.5 is now replaced by Assumption A.6 for the obvious reasons. $\triangleleft$

Remark 4.2.4 Observe that in this case the limiting variance depends on the weight function $g(\cdot)$. $\triangleleft$

4.3 Some Interesting Weight Functions

Example 4.3.1 The constant weight function $g_0(t) \equiv 6$, which clearly satisfies Assumption A.6, yields an estimator with $\mathrm{E}(C_1(g_0, n)) = \sigma^2 + 8\gamma/n + \mathrm{o}(1/n)$ and $\mathrm{Var}(C_1(g_0, n)) = 4\sigma^4/5 \approx 0.8\sigma^4$. Observe that this estimator has larger bias than the usual (level-0) CvM estimator (see Example 1.2.11). $\triangleleft$

Ideally, we would like to choose a weight function that minimizes the limiting variance of the folded weighted CvM estimator for σ^2 while satisfying the first-order unbiasedness and normalizing constraints. That is, we would like to identify a function $g(t)$ that minimizes $\mathrm{Var}(C_1(g))$ subject to $I = 0$ and the normalizing condition in Assumption A.6. We present a few such polynomial weights $g(t)$ of degree m. We do that for $m = 2, 4, 6$, to be able to make a fair comparison with the regular (level-0) weighted CvM estimator for σ^2.

Example 4.3.2 The weight function $g_{(2)}(t) = -180t^2 + 168t - 24$ makes $I = 0$ in Theorem 4.2.1. Therefore, it yields a first-order unbiased estimator with $\mathrm{Var}(C_1(g_{(2)})) = 72\sigma^4/35 \approx 2.06\sigma^4$. However, the limiting variance of this estimator is not smaller than the limiting variance of the analogous (same degree) polynomial for the regular (level-0) weighted CvM estimator, which equals $1.73\sigma^4$ (see Example 1.2.12). $\triangleleft$

Example 4.3.3 Consider the fourth degree polynomial

$$g_{(4)}(t) = at^4 + bt^3 + ct^2 + dt + e,$$

where a, b, c, d, e are real constants given by

$$a = -1050 - \frac{35}{4}d - 105e, \quad b = 1320 + 15d + 160e, \quad \text{and} \quad c = -360 - \frac{15}{2}d - 60e.$$

This weight function satisfies $G^* = 1$, $I = 0$ and Assumption A.6. As a result, it yields a first-order unbiased estimator for each choice of the real constants a, b, c, d, e. Moreover, one can show that the limiting variance of these estimators is minimized for $d = 2840/3$ and $e = -60$, with a value of $2360\sigma^4/2079 \approx 1.14\sigma^4$. That is,

$$g^*_{(4)}(t) = -60 + 2840t/3 - 3860t^2 + 5920t^3 - 9100t^4/3.$$

Again, this variance is larger than its analogue for the original weighted CvM estimator, which is equal to $1.04\sigma^4$—see Example 1.2.13. $\triangleleft$

Example 4.3.4 Similarly, we get a sixth degree polynomial weight function yielding a first-order unbiased estimator:

$$\begin{aligned} g^*_{(6)}(t) = &-\frac{41510}{327} + \frac{375760}{109}t - \frac{2982490}{109}t^2 + \frac{31856440}{327}t^3 \\ &- \frac{18914350}{109}t^4 + \frac{16444960}{109}t^5 - \frac{5547780}{109}t^6. \end{aligned}$$

The resulting weighted folded CvM estimator has limiting variance equal to $0.85\sigma^4$. Once more, this variance is larger than the limiting variance of the corresponding level-0 weighted CvM estimator; the latter has value $0.81\sigma^4$—see Example 1.2.13. $\triangleleft$

Remark 4.3.5 In order to achieve further variance reductions, we can continue to increase the degree of the polynomial weight function. However, the magnitudes of the resulting coefficients become quite large, and one must be careful to avoid round-off errors as well as deleterious second-order effects for small sample sizes. A way to deal with the magnitude problem could be to use higher precision. $\triangleleft$

4.4 Covariance Between Two Folded Estimators

Even though "folding" seems to slightly increase the variance of the CvM estimators for σ^2, the folded CvM estimators still have some advantage over some of their competitors. In fact, we can take linear combinations of different levels of CvM estimators with the ultimate goal of producing new estimators with comparatively smaller variance than before, while maintaining the same low bias. Theorem 4.4.1 computes the covariance between the limiting functionals of the level 0 and 1 CvM estimators, with the final objective of studying the asymptotic behavior of the variance of the linearly combined estimators described above. In particular, we are interested in the average of the first two levels of the weighted CvM estimators for σ^2. At the end of this section, we will compute the variances of the these combined estimators for the weight functions from Examples 1.2.11–1.2.13 and 4.3.1–4.3.4.

Theorem 4.4.1 *Let $h_i(\cdot)$ be a function satisfying Assumption A.6 for $i = 0, 1$. Then*

$$\begin{aligned}\mathrm{Cov}(C_0(h_0), C_1(h_1)) = 2\sigma^4 &\int_0^1 \int_0^{s/2} h_0(t)h_1(s)t^2(1-s)^2\, dt\, ds \\ &+ 2\sigma^4 \int_0^1 \int_{s/2}^{1-s/2} h_0(t)h_1(s)s^2 \left(\frac{1}{2} - t\right)^2 dt\, ds \\ &+ 2\sigma^4 \int_0^1 \int_{1-s/2}^{1} h_0(t)h_1(s)(1-s)^2(1-t)^2\, dt\, ds.\end{aligned}$$

Proof: By Definition 4.1.2, we have

$$\begin{aligned}&\mathrm{Cov}(C_0(h_0), C_1(h_1)) \\ &= \mathrm{Cov}\left(\int_0^1 h_0(t)[\sigma\mathcal{B}_0(t)]^2\, dt, \int_0^1 h_1(s)[\sigma\mathcal{B}_1(s)]^2\, ds\right) \\ &= \sigma^4 \int_0^1 \int_0^1 h_0(t)h_1(s)\mathrm{Cov}(\mathcal{B}_0^2(t), \mathcal{B}_1^2(s))\, dt\, ds \\ &\qquad \text{(by Fubini's Theorem and the continuity of the integrand on } [0,1]^2\text{)} \\ &= 2\sigma^4 \int_0^1 \int_0^1 h_0(t)h_1(s)\mathrm{Cov}^2(\mathcal{B}_0(t), \mathcal{B}_1(s))\, dt\, ds \\ &\qquad \text{(since } \mathcal{B}_0(\cdot) \text{ is a Gaussian process with zero mean, see Patel and Read [36])} \\ &= 2\sigma^4 \int_0^1 \int_0^{s/2} h_0(t)h_1(s)t^2(1-s)^2\, dt\, ds \\ &\quad + 2\sigma^4 \int_0^1 \int_{s/2}^{1-s/2} h_0(t)h_1(s)s^2 \left(\frac{1}{2} - t\right)^2 dt\, ds \\ &\quad + 2\sigma^4 \int_0^1 \int_{1-s/2}^{1} h_0(t)h_1(s)(1-s)^2(1-t)^2\, dt\, ds \quad \text{(by Equation (3.4.2)).} \quad \square\end{aligned}$$

Example 4.4.2 Consider the constant weight function $g_0(t) = 6$ for $0 \leq t \leq 1$ from Examples 1.2.11 and 4.3.1, and define

$$\overline{C}_{0,1}(g_0, n) \equiv \frac{C_0(g_0, n) + C_1(g_0, n)}{2}, \tag{4.4.1}$$

and

$$\overline{C}_{0,1}(g_0) \equiv \frac{C_0(g_0) + C_1(g_0)}{2}. \tag{4.4.2}$$

In this case, Theorem 4.4.1 implies

$$\mathrm{Cov}(C_0(g_0), C_1(g_0)) = \frac{\sigma^4}{5} = 0.2\sigma^4.$$

As a result, the limiting variance of $\overline{C}_{0,1}(g_0, n)$ is

$$\begin{aligned}\mathrm{Var}(\overline{C}_{0,1}(g_0)) &= \mathrm{Var}\left[\frac{C_0(g_0) + C_1(g_0)}{2}\right] \\ &= \frac{1}{4}\big[\mathrm{Var}(C_0(g_0)) + \mathrm{Var}(C_1(g_0))\big] + \frac{1}{2}\mathrm{Cov}(C_0(g_0), C_1(g_0)) \\ &= \frac{4\sigma^4}{4\cdot 5} + \frac{4\sigma^4}{4\cdot 5} + \frac{1}{2}\frac{\sigma^4}{5} = \frac{\sigma^4}{2} = 0.5\sigma^4.\end{aligned}$$

Then

$$\frac{\mathrm{Var}(\overline{C}_{0,1}(g_0))}{\mathrm{Var}(C_0(g_0))} = \frac{\mathrm{Var}(\overline{C}_{0,1}(g_0))}{\mathrm{Var}(C_1(g_0))} = \frac{5}{8} = 0.625,$$

which can be translated as a 37.5% reduction in the limiting variance. However, the estimator $\overline{C}_{0,1}(g_0, n)$ is not first-order unbiased. Indeed, Examples 1.2.11 and 4.3.1 imply that

$$\mathrm{E}(\overline{C}_{0,1}(g_0, n)) = \sigma^2 + \frac{13\gamma}{2} + \mathrm{o}(1/n). \quad \triangleleft$$

Example 4.4.3 Consider the weight functions $g_2^*(\cdot)$ and $g_{(2)}^*(\cdot)$ from Examples 1.2.12 and 4.3.2 respectively. Now define

$$\overline{C}_{0,1}(g_2^*, g_{(2)}^*, n) \equiv \frac{C_0(g_2^*, n) + C_1(g_{(2)}^*, n)}{2}, \tag{4.4.3}$$

and

$$\overline{C}_{0,1}(g_2^*, g_{(2)}^*) \equiv \frac{C_0(g_2^*) + C_1(g_{(2)}^*)}{2}. \tag{4.4.4}$$

In this case, Theorem 4.4.1 implies

$$\mathrm{Cov}(C_0(g_2^*), C_1(g_{(2)}^*)) = \frac{3\sigma^4}{28} \approx 0.11\sigma^4.$$

As a result, the limiting variance of $\overline{C}_{0,1}(g_2^*, g_{(2)}^*, n)$ is

$$\begin{aligned}\mathrm{Var}(\overline{C}_{0,1}(g_2^*, g_{(2)}^*)) &= \mathrm{Var}\left[\frac{C_0(g_2) + C_1(g_{(2)})}{2}\right] \\ &= \frac{1}{4}\big[\mathrm{Var}(C_0(g_2^*)) + \mathrm{Var}(C_1(g_{(2)}^*))\big] + \frac{1}{2}\mathrm{Cov}(C_0(g_2^*), C_1(g_{(2)}^*)) \\ &= \frac{121\sigma^4}{4\cdot 70} + \frac{72\sigma^4}{4\cdot 35} + \frac{3\sigma^4}{56} = \sigma^4.\end{aligned}$$

Observe that

$$\frac{\mathrm{Var}(\overline{C}_{0,1}(g_2^*, g_{(2)}^*))}{\mathrm{Var}(C_0(g_2^*))} = \frac{70}{121} \approx 0.58,$$

which can be translated as a 42% reduction in the limiting variance in comparison to the estimator $C_0(g_2^*, n)$. Similarly,

$$\frac{\mathrm{Var}(\overline{C}_{0,1}(g_2^*, g_{(2)}^*))}{\mathrm{Var}(C_1(g_{(2)}^*))} = \frac{35}{72} \approx 0.49,$$

which can be translated as a 51% reduction in the limiting variance in comparison to the estimator $C_1(g_{(2)}^*, n)$. Moreover, $\overline{C}_{0,1}(g_2^*, g_{(2)}^*, n)$ is still first-order unbiased. Clearly, $\overline{C}_{0,1}(g_2^*, g_{(2)}^*, n)$ outperforms the original estimators $C_0(g_2^*, n)$ and $C_1(g_{(2)}^*, n)$. $\triangleleft$

Example 4.4.4 Consider the weight functions $g_4^*(\cdot)$ and $g_{(4)}^*(\cdot)$ from Examples 1.2.13 and 4.3.3 respectively. Define

$$\overline{C}_{0,1}(g_4^*, g_{(4)}^*, n) \equiv \frac{C_0(g_4^*, n) + C_1(g_{(4)}^*, n)}{2}, \tag{4.4.5}$$

and

$$\overline{C}_{0,1}(g_4^*, g_{(4)}^*) \equiv \frac{C_0(g_4^*) + C_1(g_{(4)}^*)}{2}. \tag{4.4.6}$$

In this case, Theorem 4.4.1 implies

$$\mathrm{Cov}(C_0(g_4^*), C_1(g_{(4)}^*)) = \frac{71219\sigma^4}{254016} \approx 0.28\sigma^4.$$

As a result, the limiting variance of $\overline{C}_{0,1}(g_4^*, g_{(4)}^*, n)$ is

$$\begin{aligned}
\mathrm{Var}(\overline{C}_{0,1}(g_4^*, g_{(4)}^*)) &= \mathrm{Var}\left[\frac{C_0(g_4) + C_1(g_{(4)})}{2}\right] \\
&= \frac{1}{4}\left[\mathrm{Var}(C_0(g_4^*)) + \mathrm{Var}(C_1(g_{(4)}^*))\right] + \frac{1}{2}\mathrm{Cov}(C_0(g_4^*), C_1(g_{(4)}^*)) \\
&= \frac{1.042\sigma^4 + 1.135\sigma^4}{4} + 0.14\sigma^4 \approx 0.68\sigma^4.
\end{aligned}$$

Hence

$$\frac{\mathrm{Var}(\overline{C}_{0,1}(g_4^*, g_{(4)}^*))}{\mathrm{Var}(C_0(g_4^*))} \approx 0.65,$$

which can be translated as a 35% reduction in the limiting variance in comparison to the estimator $C_0(g_4^*, n)$. Similarly,

$$\frac{\mathrm{Var}(\overline{C}_{0,1}(g_4^*, g_{(4)}^*))}{\mathrm{Var}(C_1(g_{(4)}^*))} \approx 0.60,$$

which can be translated as a 40% reduction in the limiting variance in comparison to the estimator $C_1(g^*_{(4)}, n)$. Moreover, $\overline{C}_{0,1}(g^*_4, g^*_{(4)}, n)$ is still first-order unbiased. Clearly, $\overline{C}_{0,1}(g^*_4, g^*_{(4)}, n)$ outperforms the original estimators $C_0(g^*_4, n)$ and $C_1(g^*_{(4)}, n)$. $\triangleleft$

Example 4.4.5 Consider the weight functions $g^*_6(\cdot)$ and $g^*_{(6)}(\cdot)$ from Examples 1.2.13 and 4.3.4 respectively, and define

$$\overline{C}_{0,1}(g^*_6, g^*_{(6)}, n) \equiv \frac{C_0(g^*_6, n) + C_1(g^*_{(6)}, n)}{2}, \tag{4.4.7}$$

and

$$\overline{C}_{0,1}(g^*_6, g^*_{(6)}) \equiv \frac{C_0(g^*_6) + C_1(g^*_{(6)})}{2}. \tag{4.4.8}$$

In this case, Theorem 4.4.1 implies

$$\mathrm{Cov}(C_0(g^*_6), C_1(g^*_{(6)})) \approx 0.20\sigma^4.$$

As a result, the limiting variance of $\overline{C}_{0,1}(g^*_6, g^*_{(6)}, n)$ is

$$\begin{aligned}
\mathrm{Var}(\overline{C}_{0,1}(g^*_6, g^*_{(6)})) &= \mathrm{Var}\left[\frac{C_0(g_6) + C_1(g_{(6)})}{2}\right] \\
&= \frac{1}{4}\left[\mathrm{Var}(C_0(g^*_6)) + \mathrm{Var}(C_1(g^*_{(6)}))\right] + \frac{1}{2}\mathrm{Cov}(C_0(g^*_6), C_1(g^*_{(6)})) \\
&= \frac{0.85\sigma^4 + 0.81\sigma^4}{4} + 0.10\sigma^4 \approx 0.52\sigma^4.
\end{aligned}$$

Observe that

$$\frac{\mathrm{Var}(\overline{C}_{0,1}(g^*_6, g^*_{(6)}))}{\mathrm{Var}(C_0(g^*_6))} \approx 0.64,$$

which can be translated as a 39% reduction in the limiting variance in comparison to the estimator $C_0(g^*_6, n)$. Similarly,

$$\frac{\mathrm{Var}(\overline{C}_{0,1}(g^*_6, g^*_{(6)}))}{\mathrm{Var}(C_1(g^*_{(6)}))} \approx 0.61,$$

which can be translated as a 36% reduction in the limiting variance in comparison to the estimator $C_1(g^*_{(6)}, n)$. Moreover, $\overline{C}_{0,1}(g^*_6, g^*_{(6)}, n)$ is still first-order unbiased. Clearly, $\overline{C}_{0,1}(g^*_6, g^*_{(6)}, n)$ outperforms the original estimators $C_0(g^*_6, n)$ and $C_1(g^*_{(6)}, n)$. $\triangleleft$

CHAPTER V

ANALYTICAL EXAMPLES AND MONTE CARLO SIMULATIONS

In this chapter, we support our theoretical results first with an analytical example using the first-order moving average [MA(1)] process, and then we resort to Monte Carlo simulation to empirically evaluate the performance characteristics of the various estimators on more-complicated stochastic processes.

5.1 Some Analytical Examples

This section presents exact analytical results involving the MA(1) process. We shall first obtain some useful expressions for the expected values and variances of the folded area and folded CvM estimators. We assume in the sequel that Assumptions A are still in effect. We begin with an intermediate result on the folded area estimator.

$$\begin{aligned}
\mathrm{E}(A_1(w,n)) &= \mathrm{E}(N_1^2(w,n)) \\
&= \mathrm{Var}(N_1(w,n)) \\
&= \mathrm{Cov}\left[N_1(w,n), N_1(w,n)\right] \\
&= \mathrm{Cov}\left[\frac{1}{n}\sum_{j=1}^{n}\sigma w\left(\frac{j}{n}\right)T_{1,n}\left(\frac{j}{n}\right), \frac{1}{n}\sum_{k=1}^{n}\sigma w\left(\frac{k}{n}\right)T_{1,n}\left(\frac{k}{n}\right)\right] \\
&= \frac{\sigma^2}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)\mathrm{Cov}\left(T_{1,n}\left(\frac{j}{n}\right), T_{1,n}\left(\frac{k}{n}\right)\right), \qquad (5.1.1)
\end{aligned}$$

where

$$\begin{aligned}
&n\sigma^2\mathrm{Cov}\left(T_{1,n}\left(\frac{j}{n}\right), T_{1,n}\left(\frac{k}{n}\right)\right) \\
&= \mathrm{Cov}\left(\left(\frac{j-n}{n}\right)Z_n - Z_{\lfloor \frac{j}{2}\rfloor} + Z_{\lfloor n-\frac{j}{2}\rfloor}, \left(\frac{k-n}{n}\right)Z_n - Z_{\lfloor \frac{k}{2}\rfloor} + Z_{\lfloor n-\frac{k}{2}\rfloor}\right)
\end{aligned}$$

by Lemma 2.3.2, where $Z_n = \sum_{i=1}^{n} X_i$

$$
\begin{aligned}
&= \left(\frac{j-n}{n}\right)\left(\frac{k-n}{n}\right)\operatorname{Var}(Z_n) - \left(\frac{j-n}{n}\right)\operatorname{Cov}\left(Z_n, Z_{\lfloor \frac{k}{2} \rfloor}\right) \\
&\quad + \left(\frac{j-n}{n}\right)\operatorname{Cov}\left(Z_n, Z_{\lfloor n-\frac{k}{2} \rfloor}\right) - \left(\frac{k-n}{n}\right)\operatorname{Cov}\left(Z_{\lfloor \frac{j}{2} \rfloor}, Z_n\right) \\
&\quad + \operatorname{Cov}\left(Z_{\lfloor \frac{j}{2} \rfloor}, Z_{\lfloor \frac{k}{2} \rfloor}\right) - \operatorname{Cov}\left(Z_{\lfloor \frac{j}{2} \rfloor}, Z_{\lfloor n-\frac{k}{2} \rfloor}\right) \\
&\quad + \left(\frac{k-n}{n}\right)\operatorname{Cov}\left(Z_{\lfloor n-\frac{j}{2} \rfloor}, Z_n\right) - \operatorname{Cov}\left(Z_{\lfloor n-\frac{j}{2} \rfloor}, Z_{\lfloor \frac{k}{2} \rfloor}\right) + \operatorname{Cov}\left(Z_{\lfloor n-\frac{j}{2} \rfloor}, Z_{\lfloor n-\frac{k}{2} \rfloor}\right).
\end{aligned} \tag{5.1.2}
$$

Replacing Equation (5.1.2) into Equation (5.1.1) and collecting similar terms, we get

$$
\begin{aligned}
\mathrm{E}(A_1(w,n)) = \frac{1}{n^3}&\left[\sum_{k=1}^{n}\left[\frac{k}{n}-1\right] w\left(\frac{k}{n}\right)\right]^2 \operatorname{Var}(Z_n) \\
&- \frac{2}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{j}{n}-1\right] w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)\operatorname{Cov}\left[Z_n, Z_{\lfloor \frac{k}{2} \rfloor}\right] \\
&+ \frac{2}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{j}{n}-1\right] w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)\operatorname{Cov}\left[Z_n, Z_{\lfloor n-\frac{k}{2} \rfloor}\right] \\
&- \frac{2}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)\operatorname{Cov}\left[Z_{\lfloor \frac{j}{2} \rfloor}, Z_{\lfloor n-\frac{k}{2} \rfloor}\right] \\
&+ \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)\operatorname{Cov}\left[Z_{\lfloor \frac{j}{2} \rfloor}, Z_{\lfloor \frac{k}{2} \rfloor}\right] \\
&+ \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)\operatorname{Cov}\left[Z_{\lfloor n-\frac{j}{2} \rfloor}, Z_{\lfloor n-\frac{k}{2} \rfloor}\right].
\end{aligned} \tag{5.1.3}
$$

Similarly, we get an intermediate result for the folded CvM estimator,

$$
\begin{aligned}
&\mathrm{E}(C_1(g,n)) \\
&= \mathrm{E}\left(\frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\sigma T_{1,n}\left(\frac{k}{n}\right)\right]^2\right) \\
&= \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\mathrm{E}\left(\frac{k-n}{n}Z_n - Z_{\lfloor \frac{k}{2} \rfloor} + Z_{\lfloor n-\frac{k}{2} \rfloor}\right)^2 \\
&= \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\frac{k-n}{n}\right]^2 \mathrm{E}\left(Z_n^2\right) + \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\mathrm{E}\left(Z_{\lfloor \frac{k}{2} \rfloor}^2\right) \\
&\quad + \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\mathrm{E}\left(Z_{\lfloor n-\frac{k}{2} \rfloor}^2\right) - \frac{2}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\frac{k-n}{n}\right]\mathrm{E}\left(Z_n Z_{\lfloor \frac{k}{2} \rfloor}\right) \\
&\quad + \frac{2}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\frac{k-n}{n}\right]\mathrm{E}\left(Z_n Z_{\lfloor n-\frac{k}{2} \rfloor}\right) - \frac{2}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\mathrm{E}\left(Z_{\lfloor \frac{k}{n} \rfloor} Z_{\lfloor n-\frac{k}{2} \rfloor}\right).
\end{aligned} \tag{5.1.4}
$$

We now have at our disposal the machinery to study specific examples in which we calculate the exact expected values of $A_1(w,n)$ and $C_1(g,n)$ for various weight functions.

For the remainder of this section, we shall work with the MA(1) process, $X_{i+1} = \theta\epsilon_i + \epsilon_{i+1}$, $i = 1, 2, \ldots$, where the ϵ_i's are i.i.d. Nor(0,1); thus, $R_0 = 1+\theta^2$, $R_{\pm 1} = \theta$, and $R_k = 0$, otherwise. One can derive from Lemma B.1.1

$$\mathrm{Var}(Z_n) = n(1+\theta)^2 - 2\theta = n\sigma^2 + \gamma \quad \text{for all } n \geq 1, \tag{5.1.5}$$

and from Lemma B.1.3,

$$\mathrm{Cov}(Z_n, Z_k) = (1+\theta)^2 k - \theta = k\sigma^2 + \frac{\gamma}{2} \quad \text{for } k < n, \tag{5.1.6}$$

where $\sigma^2 = (1+\theta)^2$ and $\gamma = -2\theta$. Using Lemma B.1.5 we get

$$\mathrm{E}(Z_n Z_k) = (1+\theta)^2 k - \theta + kn\mu^2 = k\sigma^2 + \frac{\gamma}{2} + kn\mu^2 \quad \text{for } k < n, \tag{5.1.7}$$

and from Equation (5.1.5) we find

$$\mathrm{E}(Z_n^2) = n(1+\theta)^2 - 2\theta + n^2\mu^2 = n\sigma^2 + \gamma + n^2\mu^2 \quad \text{for all } n \geq 1. \tag{5.1.8}$$

Now, plugging Equations (5.1.5), (5.1.6) into Equation (5.1.3),

$$\begin{aligned}
\mathrm{E}(A_1(w,n)) = {} & \frac{1}{n^3}\left[\sum_{k=1}^{n}\left[\frac{k}{n}-1\right] w\left(\frac{k}{n}\right)\right]^2 (n\sigma^2+\gamma) \\
& \text{(by Equation (5.1.5))} \\
& - \frac{2}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{j}{n}-1\right] w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) (\left\lfloor\frac{k}{2}\right\rfloor\sigma^2 + \frac{\gamma}{2}) \\
& \text{(by Equation (5.1.6) and } \lfloor k/2 \rfloor \leq n) \\
& + \frac{2}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{j}{n}-1\right] w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) (\left\lfloor n-\frac{k}{2}\right\rfloor\sigma^2 + \frac{\gamma}{2}) \\
& \text{(by Equation (5.1.6) and } \lfloor n-k/2 \rfloor \leq n) \\
& - \frac{2}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) (\left\lfloor\frac{j}{2}\right\rfloor\sigma^2 + \frac{\gamma}{2}) \\
& \text{(by Equation (5.1.6) and } \lfloor j/2 \rfloor \leq \lfloor n-k/2 \rfloor)
\end{aligned}$$

$$+ \frac{1}{n^3} \sum_{j=1}^{n} \sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) \left\{ \left(\left\lfloor \frac{j}{2} \right\rfloor \wedge \left\lfloor \frac{k}{2} \right\rfloor \right) \sigma^2 + \frac{\gamma}{2} \right\}$$

$$+ \frac{1}{n^3} \sum_{j=1}^{n} \sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) \left\{ \left(\left\lfloor n - \frac{j}{2} \right\rfloor \wedge \left\lfloor n - \frac{k}{2} \right\rfloor \right) \sigma^2 + \frac{\gamma}{2} \right\}, \quad (5.1.9)$$

by Equation (5.1.6). Now, by grouping similar terms we get

$$\begin{aligned}
\mathrm{E}(A_1(w,n)) = {} & \frac{\gamma}{n^3} \left[\sum_{k=1}^{n} \left[\frac{k}{n} - 1 \right] w\left(\frac{k}{n}\right) \right]^2 \\
& - \frac{\sigma^2}{n^2} \left[\sum_{k=1}^{n} \left[\frac{k}{n} - 1 \right] w\left(\frac{k}{n}\right) \right]^2 \\
& - \frac{2\sigma^2}{n^3} \sum_{j=1}^{n} \left\lfloor \frac{j}{2} \right\rfloor w\left(\frac{j}{n}\right) \sum_{k=1}^{n} w\left(\frac{k}{n}\right) \\
& + \frac{\sigma^2}{n^3} \sum_{j=1}^{n} \sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) \left(\left\lfloor \frac{j}{2} \right\rfloor \wedge \left\lfloor \frac{k}{2} \right\rfloor \right) \\
& + \frac{\sigma^2}{n^3} \sum_{j=1}^{n} \sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) \left(\left\lfloor n - \frac{j}{2} \right\rfloor \wedge \left\lfloor n - \frac{k}{2} \right\rfloor \right). \quad (5.1.10)
\end{aligned}$$

Example 5.1.1 If we take $w_0(t) = \sqrt{12}$, we get that

$$\begin{aligned}
\left[\sum_{k=1}^{n} \left[\frac{k}{n} - 1 \right] w\left(\frac{k}{n}\right) \right]^2 &= 3(1-n)^2 \\
\sum_{j=1}^{n} \left\lfloor \frac{j}{2} \right\rfloor w\left(\frac{j}{n}\right) &= \frac{\sqrt{3}n^2}{2} \\
\sum_{k=1}^{n} w\left(\frac{k}{n}\right) &= 2\sqrt{3}n \\
\sum_{j=1}^{n} \sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) \left(\left\lfloor \frac{j}{2} \right\rfloor \wedge \left\lfloor \frac{k}{2} \right\rfloor \right) &= 2n(n^2-1) \\
\sum_{j=1}^{n} \sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) \left(\left\lfloor n - \frac{j}{2} \right\rfloor \wedge \left\lfloor n - \frac{k}{2} \right\rfloor \right) &= 2n(4n^2 - 3n + 2)
\end{aligned}$$

Plugging these sums into Equation (5.1.10), we get

$$\begin{aligned}
\mathrm{E}(A_1(w_0,n)) &= \left(1 - \frac{1}{n^2}\right) \sigma^2 + \frac{3(1-n)^2\gamma}{n^3} \\
&= \sigma^2 + \frac{3}{n}\gamma + \mathrm{o}(1/n). \quad \triangleleft
\end{aligned}$$

Remark 5.1.2 We just computed, by hand, $W^*_{D,n} = 1 - 1/n^2$, the exact value of the coefficient of σ^2 in $\mathrm{E}(A_1(w_0,n))$ for the MA(1) process and the w_0 weight—see Definition 3.2.4. We could have obtained the same result by using Equation (B.3.1) from Appendix B.

Contrast this with the value of $W_n^* = 1 + 6/n$ given by Theorem 3.2.3 in Example 3.3.1 and remember that in the computation of W_n^* multiple terms were discarded because of their order, while in $W_{D,n}^*$ preserved explicitly all those terms $\triangleleft$

For the folded CvM estimator, plugging Equations (5.1.7) and (5.1.8) into Equation (5.1.4), we get

$$\begin{aligned}
\mathrm{E}(C_1(g,n)) &= \mathrm{E}\left(\frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\sigma T_1\left(\frac{k}{n}\right)\right]^2\right)\\
&= \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\mathrm{E}\left(\frac{k-n}{n}Z_n - Z_{\lfloor\frac{k}{2}\rfloor} + Z_{\lfloor n-\frac{k}{2}\rfloor}\right)^2\\
&= \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\frac{k-n}{n}\right]^2 (n\sigma^2 + \gamma + n^2\mu^2)\\
&\quad + \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left(\left\lfloor\frac{k}{2}\right\rfloor\sigma^2 + \gamma + \left\lfloor\frac{k}{2}\right\rfloor^2\mu^2\right)\\
&\quad + \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left(\left\lfloor n-\frac{k}{2}\right\rfloor\sigma^2 + \gamma + \left\lfloor n-\frac{k}{2}\right\rfloor^2\mu^2\right)\\
&\quad - \frac{2}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\frac{k-n}{n}\right]\left(\left\lfloor\frac{k}{2}\right\rfloor\sigma^2 + \frac{\gamma}{2} + n\left\lfloor\frac{k}{2}\right\rfloor\mu^2\right)\\
&\quad + \frac{2}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\frac{k-n}{n}\right]\left(\left\lfloor n-\frac{k}{2}\right\rfloor\sigma^2 + \frac{\gamma}{2} + n\left\lfloor n-\frac{k}{2}\right\rfloor\mu^2\right)\\
&\quad - \frac{2}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left(\left\lfloor\frac{k}{2}\right\rfloor\sigma^2 + \frac{\gamma}{2} + \left\lfloor n-\frac{k}{2}\right\rfloor\left\lfloor\frac{k}{2}\right\rfloor\mu^2\right).
\end{aligned} \tag{5.1.11}$$

Now, grouping similar terms we get

$$\begin{aligned}
\mathrm{E}(C_1(g,n)) = &\left[\frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)(n-k) - \frac{1}{n^3}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)(k-n)^2\right]\sigma^2\\
&+ \left[\frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left(\frac{k-n}{n}\right)^2 + \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\right]\gamma.
\end{aligned}$$

Example 5.1.3 If we consider $g_0(t) = 6$, we get that

$$\begin{aligned}
\mathrm{E}(C_1(g_0,n)) &= \left(1 - \frac{1}{n^2}\right)\sigma^2 + \left(\frac{8n^2 - 3n + 1}{n^3}\right)\gamma\\
&= \sigma^2 + \frac{8}{n}\gamma + \mathrm{o}(1/n)
\end{aligned}$$

since

$$\frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left(\frac{k-n}{n}\right)^2 = \frac{(n-1)(2n-1)}{n^3},$$
$$\frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left(\frac{n-k}{n}\right) = -\frac{3(n-1)}{n^2},$$
$$\frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right) = \frac{6}{n}.$$

Again, let us recall that $\mathrm{E}(C_1(g_0,n))$ computed in Theorem 4.2.1 is an approximation. We computed the exact value for the MA(1) process in this example.

Remark 5.1.4 Notice that the MA(1) process supports our theoretical results from Theorems 3.2.3 and 4.2.1. $\triangleleft$

We resort to Monte Carlo simulation in the next section to empirically evaluate the performance characteristics of the various estimators on more-complicated stochastic processes.

5.2 Empirical Examples

In this section, we present empirical examples illustrating the performance characteristics of the following estimators for σ^2:

- $A_i(w_0,n)$: unweighted level-i area estimator for $i=0,1$.
- $\overline{A}_{0,1}(w_0,n) \equiv (A_0(w_0,n)+A_1(w_0,n))/2$.
- $A_0(w_{\cos,1},n)$: weighted level-0 area estimator.
- $A_1(w_{\sin,1},n)$: weighted level-1 area estimator.
- $\overline{A}_{0,1}(w_{\cos,1},w_{\sin,1},n) \equiv (A_0(w_{\cos,1},n)+A_1(w_{\sin,1},n))/2$.
- $C_i(g_0,n)$: unweighted level-i CvM estimator for $i=0,1$.
- $\overline{C}_{0,1}(g_0,n) \equiv (C_0(g_0,n)+C_1(g_0,n))/2$.
- $C_0(g_j^*,n)$: level-0 degree-j CvM estimator for $j=2,4,6$.
- $C_1(g_{(j)}^*,n)$: level-1 degree-j CvM estimator for $j=2,4,6$.

- $\overline{C}_{0,1}(g_j^*, g_{(j)}^*, n) \equiv (C_0(g_j^*, n) + C_1(g_{(j)}^*, n))/2$ for $j = 2, 4, 6$.

Remember that $A_0(w_{\cos,1}, n)$ and $A_1(w_{\sin,1}, n)$ are first-order unbiased and note that $C_0(g_j^*, n)$ and $C_1(g_{(j)}^*, n)$ are first-order unbiased with minimum variance for their respective polynomial order for all $j = 2, 4, 6$. Also, observe that we are going to perform these experiments using just one large batch of size n. In Chapter VI, we will perform similar experiments, but we will be applying batching techniques to the estimators.

The foregoing involve the Monte Carlo simulation of a number of stationary stochastic processes:

- The first-order autoregressive [AR(1)] process, $X_{i+1} = \varphi X_i + \epsilon_{i+1}$ for $i = 1, 2, \ldots$, where the ϵ_i's are i.i.d. $\mathrm{Nor}(0, 1 - \varphi^2)$ with $-1 < \varphi < 1$.

- The M/M/1 queueing system's waiting-time process.

For the AR(1) process, the covariance function is $R_k = \varphi^{|k|}, \quad k = 0, \pm 1, \pm 2, \ldots$, the variance parameter is $\sigma^2 = (1+\varphi)/(1-\varphi)$ (see Moran [35] or Sargent, Kang and Goldsman [39]) and $\gamma = -2\varphi/(1-\varphi)^2$ (see Sargent et al. [39]). The covariance function of the M/M/1 waiting time process is more complicated (cf. Daley [13]), but Whitt [47] (Equation (24)) finds that for traffic intensity ρ, we have $\sigma^2 = \rho(2 + 5\rho - 4\rho^2 + \rho^3)/(1-\rho)^4$ whenever the service rate equals 1. If the service rate is arbitrary, but still greater than the arrival rate λ, then Steiger and Wilson [45, p. 287] provide us with the following expression $\sigma^2 = \rho^3(2 + 5\rho - 4\rho^2 + \rho^3)/[(1-\rho)^4\lambda^2]$.

We simulated the above stochastic processes over a variety of parameter values; representative results are presented in Tables 2 and 3 for the AR(1) process and in Table 4 for the M/M/1 process. Each table entry in a row is based on 2^{17} = 131,072 independent replications of the stochastic process, and it represents the estimated expected value of the corresponding estimator and the estimated standard error of the estimator, which is computed as the sample standard deviation of the 131,072 independent replications divided by $\sqrt{131{,}072}$. We also show different sample sizes to see the effect in the bias of increasing sample sizes. All digits given in the estimated values are significant. Each of the replications was initialized from the appropriate steady-state distribution. All uniform [normal] random variables were generated from the random number generator of L'Ecuyer [31]. Exponential deviates used inversion; the M/M/1 waiting-time process was generated from an algorithm due to Schmeiser and Song [41].

5.3 Discussion

This section summarizes and discusses the exact and estimated expectation and standard error results for the variance estimators examined in Chapters III and IV. Recall that we obtained exact results for the folded area estimators in Chapter III and for the folded CvM estimators in Chapter IV. We also gave exact results for a specific stochastic process, the MA(1) process in Section 5.1, and empirical results for the AR(1) and M/M/1 processes in Section 5.2.

In Table 1, we show the theoretical standard errors corresponding to the three stochastic processes under study. The theoretical results from Chapters III and IV and a number of replications equal to 131,072 were used to compute the values in the table. For comparison reasons with the empirical results, the entries in Table 1 have been rounded to the same precision as the entries displayed in the table corresponding to each process (Tables 2, 3, and 4).

For each of the stochastic processes under study, the expected value of the unweighted area estimator $A_0(w_0, n)$ converged relatively slowly to σ^2 as n increased. This phenomenon is due to the fact that $\text{Bias}(A_0(w_0, n)) \approx 3\gamma/n$ (see Example 1.2.6). We also observe similar behavior for the unweighted level-1 folded area estimator, this being consistent with the theoretical result found in Example 3.3.1. However, the convergence of these estimators to the true value of σ^2 seems to be even slower for the M/M/1 waiting time process than for either of the two AR(1) processes studied. In the case of the first-order unbiased weighted level-0 and level-1 area estimators, the convergence is faster than that of their unweighted counterparts. It is relevant to point out that standard error estimates for all the estimators agree with the limiting theoretical value shown in Table 1. We also observe that these estimates for the standard error of the estimators get closer to the limiting theoretical value as n increases. This shows once again the consistency of our results. Now, the real

Table 1: Theoretical Limiting Standard Errors for 131,072 replications

Estimator	Stochastic Process		
	AR(1) $\varphi = 0.9$	AR(1) $\varphi = -0.9$	M/M/1 $\rho = 0.8$
$A_0(w_0)$	0.07	0.0002	8
$A_1(w_0)$	0.07	0.0002	8
$\overline{A}_{0,1}(w_0)$	0.05	0.0001	5
$A_0(w_{\cos,1})$	0.07	0.0002	8
$A_1(w_{\sin,1})$	0.07	0.0002	8
$\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1})$	0.05	0.0001	5
$C_0(g_0)$	0.05	0.0001	5
$C_1(g_0)$	0.05	0.0001	5
$\overline{C}_{0,1}(g_0)$	0.04	0.0001	4
$C_0(g_2)$	0.07	0.0002	7
$C_1(g_{(2)})$	0.08	0.0002	8
$\overline{C}_{0,1}(g_2, g_{(2)})$	0.05	0.0001	5
$C_0(g_4)$	0.05	0.0001	6
$C_1(g_{(4)})$	0.06	0.0001	6
$\overline{C}_{0,1}(g_4, g_{(4)})$	0.04	0.0001	5
$C_0(g_6)$	0.05	0.0001	5
$C_1(g_{(6)})$	0.05	0.0001	5
$\overline{C}_{0,1}(g_6, g_{(6)})$	0.04	0.0001	4

Table 2: Estimated Expected Values and Standard Errors of Variance Estimators for an AR(1) Process with $\varphi = 0.9$.

	n			
$\sigma^2 = 19$	256	1024	4096	16384
$A_0(w_0, n)$	16.75	18.49	19.01	18.86
Standard Error	0.07	0.07	0.07	0.07
$A_1(w_0, n)$	16.52	18.57	18.91	19.10
Standard Error	0.06	0.07	0.07	0.07
$\overline{A}_{0,1}(w_0, n)$	16.63	18.53	18.96	18.98
Standard Error	0.05	0.05	0.05	0.05
Correlation	0.00	0.00	0.00	0.00
$A_0(w_{\cos,1}, n)$	18.08	18.88	18.92	18.93
Standard Error	0.07	0.07	0.07	0.07
$A_1(w_{\sin,1}, n)$	16.38	18.93	18.94	18.90
Standard Error	0.06	0.07	0.07	0.07
$\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, n)$	17.23	18.91	18.93	18.91
Standard Error	0.05	0.05	0.05	0.05
Correlation	0.00	0.00	0.00	0.00
$C_0(g_0, n)$	15.74	18.10	18.87	18.90
Standard Error	0.04	0.05	0.05	0.05
$C_1(g_0, n)$	14.15	17.63	18.66	18.96
Standard Error	0.04	0.05	0.05	0.05
$\overline{C}_{0,1}(g_0, n)$	14.94	17.87	18.77	18.93
Standard Error	0.03	0.04	0.04	0.04
Correlation	0.24	0.25	0.25	0.25
$C_0(g_2^*, n)$	18.04	18.93	18.95	18.88
Standard Error	0.06	0.07	0.07	0.07
$C_1(g_{(2)}^*, n)$	16.71	18.89	19.04	19.05
Standard Error	0.07	0.07	0.08	0.08
$\overline{C}_{0,1}(g_2^*, g_{(2)}^*, n)$	17.38	18.91	19.00	18.96
Standard Error	0.05	0.05	0.05	0.05
Correlation	0.07	0.06	0.06	0.06
$C_0(g_4^*, n)$	17.18	18.69	19.11	18.99
Standard Error	0.05	0.05	0.05	0.05
$C_1(g_{(4)}^*, n)$	15.47	18.49	18.96	19.08
Standard Error	0.05	0.05	0.06	0.06
$\overline{C}_{0,1}(g_4^*, g_{(4)}^*, n)$	16.32	18.59	19.03	19.03
Standard Error	0.04	0.04	0.04	0.04
Correlation	0.31	0.29	0.29	0.29
$C_0(g_6^*, n)$	16.05	18.60	19.01	18.99
Standard Error	0.04	0.05	0.05	0.05
$C_1(g_{(6)}^*, n)$	14.14	18.21	18.95	19.07
Standard Error	0.04	0.05	0.05	0.05
$\overline{C}_{0,1}(g_6^*, g_{(6)}^*, n)$	15.10	18.41	18.98	19.03
Standard Error	0.03	0.04	0.04	0.04
Correlation	0.31	0.30	0.30	0.31

Table 3: Estimated Expected Values and Standard Errors of Variance Estimators for an AR(1) Process with $\varphi = -0.9$.

	n			
$\sigma^2 = 0.05$	256	1024	4096	16384
$A_0(w_0, n)$	0.0585	0.0541	0.0530	0.0527
Standard Error	0.0002	0.0002	0.0002	0.0002
$A_1(w_0, n)$	0.0583	0.0541	0.0531	0.0527
Standard Error	0.0002	0.0002	0.0002	0.0002
$\overline{A}_{0,1}(w_0, n)$	0.0584	0.0541	0.0530	0.0527
Standard Error	0.0002	0.0001	0.0001	0.0001
Correlation	-0.0021	-0.0003	0.0025	0.0043
$A_0(w_{\cos,1}, n)$	0.0527	0.0523	0.0526	0.0527
Standard Error	0.0002	0.0002	0.0002	0.0002
$A_1(w_{\sin,1}, n)$	0.0530	0.0530	0.0526	0.0528
Standard Error	0.0002	0.0002	0.0002	0.0002
$\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, n)$	0.0529	0.0527	0.0526	0.0527
Standard Error	0.0001	0.0001	0.0001	0.0001
Correlation	0.0013	0.0006	-0.0016	0.0055
$C_0(g_0, n)$	0.0624	0.0551	0.0532	0.0528
Standard Error	0.0002	0.0001	0.0001	0.0001
$C_1(g_0, n)$	0.0677	0.0565	0.0536	0.0529
Standard Error	0.0001	0.0001	0.0001	0.0001
$\overline{C}_{0,1}(g_0, n)$	0.0651	0.0558	0.0534	0.0528
Standard Error	0.0001	0.0001	0.0001	0.0001
Correlation	0.2497	0.2507	0.2523	0.2537
$C_0(g_2^*, n)$	0.0523	0.0527	0.0526	0.0527
Standard Error	0.0002	0.0002	0.0002	0.0002
$C_1(g_{(2)}^*, n)$	0.0544	0.0527	0.0528	0.0525
Standard Error	0.0002	0.0002	0.0002	0.0002
$\overline{C}_{0,1}(g_2^*, g_{(2)}^*, n)$	0.0534	0.0527	0.0526	0.0526
Standard Error	0.0002	0.0001	0.0001	0.0001
Correlation	0.0494	0.0532	0.0574	0.0606
$C_0(g_4^*, n)$	0.0529	0.0525	0.0527	0.0525
Standard Error	0.0002	0.0002	0.0001	0.0001
$C_1(g_{(4)}^*, n)$	0.0567	0.0531	0.0527	0.0528
Standard Error	0.0002	0.0002	0.0002	0.0002
$\overline{C}_{0,1}(g_4^*, g_{(4)}^*, n)$	0.0548	0.0528	0.0527	0.0527
Standard Error	0.0001	0.0001	0.0001	0.0001
Correlation	0.2790	0.2817	0.2890	0.2934
$C_0(g_6^*, n)$	0.0528	0.0528	0.0527	0.0526
Standard Error	0.0001	0.0001	0.0001	0.0001
$C_1(g_{(6)}^*, n)$	0.0592	0.0534	0.0526	0.0526
Standard Error	0.0001	0.0001	0.0001	0.0001
$\overline{C}_{0,1}(g_6^*, g_{(6)}^*, n)$	0.0560	0.0510	0.0526	0.0526
Standard Error	0.0001	0.0001	0.0001	0.0001
Correlation	0.2959	0.3035	0.2977	0.3003

Table 4: Estimated Expected Values and Standard Errors of Variance Estimators for an M/M/1 Process with $\rho = 0.8$.

	n					
$\sigma^2 = 1976$	256	1024	4096	16384	65536	131072
$A_0(w_0, n)$	1003	1666	1899	1955	1970	1971
Standard Error	7	15	13	10	8	8
$A_1(w_0, n)$	709	1584	1878	1954	1955	1985
Standard Error	5	14	12	9	8	8
$\overline{A}_{0,1}(w_0, n)$	856	1625	1888	1954	1962	1978
Standard Error	5	12	11	8	6	6
Correlation	0.25	0.36	0.41	0.25	0.08	0.04
$A_0(w_{\cos,1}, n)$	1010	1801	1923	1947	1979	1984
Standard Error	8	5	12	9	8	8
$A_1(w_{\sin,1}, n)$	517	1537	1908	1984	1969	1963
Standard Error	3	14	15	11	9	7
$\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, n)$	764	1669	1916	1965	1974	1974
Standard Error	5	12	11	8	6	6
Correlation	0.23	0.26	0.23	0.14	0.05	0.02
$C_0(g_0, n)$	807	1538	1048	1937	1961	1972
Standard Error	5	11	10	7	5	5
$C_1(g_0, n)$	515	1349	1777	1920	1948	1974
Standard Error	3	9	8	6	5	5
$\overline{C}_{0,1}(g_0, n)$	661	1444	1813	1929	1955	1973
Standard Error	4	9	8	6	4	4
Correlation	0.41	0.58	0.68	0.55	0.36	0.31
$C_0(g_2^*, n)$	1060	1781	1920	1963	1978	1980
Standard Error	8	15	11	8	7	7
$C_1(g_{(2)}^*, n)$	698	1599	1919	1959	1948	1996
Standard Error	5	12	12	9	8	8
$\overline{C}_{0,1}(g_2^*, g_{(2)}^*, n)$	879	1690	1923	1961	1963	1988
Standard Error	5	12	10	7	6	6
Correlation	0.31	0.51	0.55	0.28	0.12	0.10
$C_0(g_4^*, n)$	854	1674	1931	1966	1961	1978
Standard Error	6	12	11	7	6	6
$C_1(g_{(4)}^*, n)$	562	1466	1893	1968	1954	1981
Standard Error	3	10	10	8	6	6
$\overline{C}_{0,1}(g_4^*, g_{(4)}^*, n)$	708	1570	1912	1967	1958	1979
Standard Error	4	3	10	7	5	5
Correlation	0.48	0.59	0.63	0.50	0.36	0.33
$C_0(g_6^*, n)$	761	1554	1892	1967	1962	1975
Standard Error	5	11	11	8	6	5
$C_1(g_{(6)}^*, n)$	488	1333	1825	1955	1958	1971
Standard Error	3	9	10	8	6	5
$\overline{C}_{0,1}(g_6^*, g_{(6)}^*, n)$	624	1443	1859	1961	1960	1973
Standard Error	3	9	10	7	5	4
Correlation	0.48	0.55	0.61	0.55	0.39	0.36

finding is the behavior of the estimators $\overline{A}_{0,1}(w_0, n)$ and $\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, n)$. Even though $\overline{A}_{0,1}(w_0, n)$ converges slowly to σ^2 (given its bias $\approx 3\gamma/n$), its variance is 50% smaller than the variances of $A_0(w_0, n)$ and $A_1(w_0, n)$ alone. Not only is $\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, n)$ first-order unbiased, but it also reduces the variance by 50% in comparison with the variances of the two estimators in play. Both cases are a direct consequence of the fact that the covariance between the two estimators averaged is zero as shown at the end of Chapter III.

On the other hand, we have the unweighted CvM estimators. We observe in all processes the similar behavior for both levels, even though $C_1(g_0, n)$ converges more slowly to σ^2 than $C_0(g_0, n)$ does. This is due to the fact that $\mathrm{Bias}(C_1(g_0, n)) \approx 8\gamma/n$ while $\mathrm{Bias}(C_0(g_0, n)) \approx 5\gamma/n$. Additionally, we notice the convergence of the sample standard errors to their limiting theoretical value in Table 1 as n increases. See Examples 1.2.11 and 4.3.1. When we look at the empirical results for $\overline{C}_{0,1}(g_0, n)$, we observe that the variance reduction is not as dramatic as for the analogous unweighted area estimator. This unfortunate fact is explained by the positive covariance between the two levels of unweighted CvM estimators as shown at the end of Chapter IV. However, we can see from the empirical results that the unweighted CvM variances are about 40% smaller than their unweighted area counterparts, agreeing once more with the theoretical results shown in Examples 1.2.11 and 4.3.1. Indeed,

$$\mathrm{Var}(C_i(g_0))/\mathrm{Var}(A_i(w_0)) = \frac{2}{5} = 0.4 \quad \text{for both levels } i = 0, 1.$$

If we take, for example, $\widehat{\mathrm{SSE}}(C_1(g_0, 131{,}072)) = 5$ and $\widehat{\mathrm{SSE}}(A_1(w_0, 131{,}072)) = 8$, the entries corresponding to the sample standard errors of estimators $C_1(g_0, 131{,}072)$ and $A_1(w_0, 131{,}072)$ from Table 4 (for example), we then have that

$$\frac{\widehat{\mathrm{SSE}}^2(C_1(g_0, 131{,}072))}{\widehat{\mathrm{SSE}}^2(A_1(w_0, 131{,}072))} = \frac{5^2}{8^2} \approx 0.4.$$

Within the limitations imposed on us by the selected precision, that pattern is observed acrros the tables, corroborating our theoretical results. The expected values of the minimum variance first-order unbiased level-0 quadratic CvM estimator for σ^2, $C_0(g_2^*, n)$, and the minimum variance first-order unbiased quadratric level-1 CvM estimator for σ^2, $C_1(g_{(2)}^*, n)$, converged comparatively quickly to σ^2 as n increased; the rapid convergence is a direct consequence of the first-order unbiasedness of the estimators. For large n, we see from Examples 1.2.12 and 4.3.2 that

$$\mathrm{Var}(C_0(g_2^*))/\mathrm{Var}(C_1(g_{(2)}^*)) = \frac{\frac{121\sigma^4}{70}}{\frac{72\sigma^4}{35}} = \frac{121}{144} \approx 0.84.$$

This shows how folding seems to be increasing the variance by about 16% in the quadratic case. However, the huge advantage of having two estimators for which the covariance is known is that we can average them to achieve variance reduction. This is the case for the estimator $\overline{C}_{0,1}(g_2^*, g_{(2)}^*, n)$, whose expectation converges as quickly to σ^2 as the individual first-order unbiased minimum-variance quadratic CvM estimators for σ^2; but it also achieves a variance reduction with respect to the original level-0 and level-1 estimators. The reduction is not as drastic as the 50% achieved by the area estimators (due to the positive covariance between the two estimators involved) but it is still significant by all means.

The minimum-variance first-order unbiased fourth and sixth-degree level-0 and level-1 CvM estimators, $C_0(g_4^*, n)$, $C_0(g_6^*, n)$, $C_1(g_{(4)}^*, n)$, and $C_1(g_{(6)}^*, n)$ possess expected values that converge to σ^2 almost (but not quite) as quickly as those of $C_0(g_2^*, n)$ and $C_1(g_{(2)}^*, n)$. A favorable property of these higher-degree estimators is that they have reduced standard errors. For instance, for the MA(1) process and the AR(1) process (see Tables 2 and 3), the standard error improvements for $C_0(g_4^*, n)$, $C_0(g_6^*, n)$, $C_1(g_{(4)}^*, n)$ and $C_1(g_{(6)}^*, n)$ are along the lines indicated by Examples 1.2.13, 4.3.3, and 4.3.4. Such improvements were not quite observed for the M/M/1 process with high traffic intensity for small values of n (see Table 4), but it is notable for larger run lengths. Finally, to study the effect that folding has on variance reduction for the fourth and sixth-degree cases note that

$$\mathrm{Var}(C_0(g_4^*))/\mathrm{Var}(C_1(g_{(4)}^*)) = \frac{1.042\sigma^4}{1.135\sigma^4} \approx 0.91$$

and

$$\mathrm{Var}(C_0(g_6^*))/\mathrm{Var}(C_1(g_{(6)}^*)) = \frac{0.8093\sigma^4}{0.8514\sigma^4} \approx 0.95.$$

The bottom line is that among all the estimators studied so far, the averaged ones performed similarly to their constituents in terms of expected value convergence to σ^2, but the averaged estimators outperformed the nonaveraged estimators with respect to variance.

In Figure 2, we can see the behavior of the different averaged estimators for increasing sample sizes with an AR(1) process with $\phi = 0.9$. Every plot is based on $2^{12} = 4096$ independent replications.

On the horizontal axis, $\log_2(n)$ is represented, and all powers of 2 between 2^9 and 2^{19} were used. In gray we have a reference line corresponding to $\sigma^2 = 19$ in Figure 2.a and to $\sqrt{\sigma^4/\text{replications}} \approx 0.2969$ in Figure 2.b (the value for Figure 2.b is the theoretical standard error for the area estimators).

The area estimators are in warm hues ($\overline{A}_{0,1}(w_0, n)$ in red and $\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, n)$ in orange) and the CvM estimators are in cold hues ($\overline{C}_{0,1}(g_0, n)$ in black, $\overline{C}_{0,1}(g_2^*, g_{(2)}^*, n)$ in blue, $\overline{C}_{0,1}(g_4^*, g_{(4)}^*, n)$ in greenish blue, and $\overline{C}_{0,1}(g_6^*, g_{(6)}^*, n)$ in bluish green).

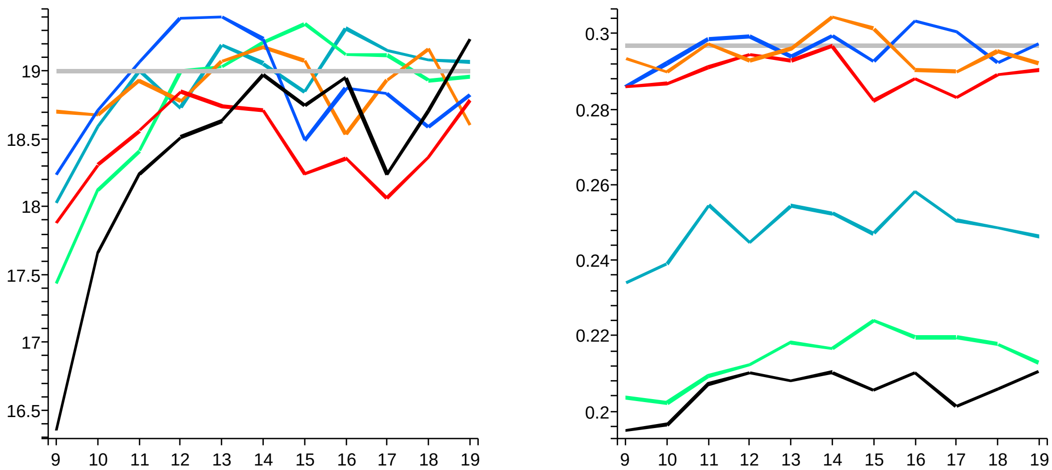
A couple of features are apparent in Figure 2:

- The behavior of the area estimators and the CvM estimator of degree 2 is similar, at least in terms of expected value convergence and standard error. See Section 6.5 Chapter VI for a more subtle insight.
- The standard error of the CvM estimators diminishes, perhaps toward the standard error of the constant weight CvM estimator as the polynomial degree increases.

5.4 Approximate Bias, Variance, and Mean Squared Error of the Folded Area Estimators

Now we can give approximate values for the bias, variance, and the mean squared error of the folded area estimators considered. We base our computations on the expected values, variances, and covariance from Theorems 3.2.3 and 3.4.1 and the computations carried out in Examples 3.3.1 and 3.3.2. Table 5 shows these results. It is important to notice that this table is valid for any stochastic process satisfying Assumptions A.1–A.4.

Table 5 confirms our empirical examples in Section 5.2 showing that $\overline{A}_{0,1}(w_0, n)$ and $\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, n)$ behave pretty much the same in terms of variance, even though $\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, n)$ is first-order unbiased and $\overline{A}_{0,1}(w_0, n)$ is not. The bottom line is that averaged estimators outperform their non-averaged counterparts.



a. Sample Average **b.** Sample Standard Error

Figure 2: Comparison of all estimators.
On the abscissas we have $\log_2(n)$, where n is the sample size. The area estimators are in warm hues and the CvM estimators are in cold hues. The line in grey is a reference value.

Table 5: Approximate Asymptotic Bias, Variance, and Mean Squared Error for Folded Area Estimators of the Variance Parameter of a Stationary Stochastic Process.

Folded Area Estimators	Approx. (n/γ)Bias	Approx. $(1/\sigma^4)$Var	Approx. MSE
$A_0(w_0, n)$	3	2	$\frac{9\gamma^2}{n^2} + 2\sigma^4$
$A_1(w_0, n)$	3	2	$\frac{9\gamma^2}{n^2} + 2\sigma^4$
$\overline{A}_{0,1}(w_0, n)$	3	1	$\frac{9\gamma^2}{n^2} + \sigma^4$
$A_0(w_{\cos,1}, n)$	0	2	$2\sigma^4$
$A_1(w_{\sin,1}, n)$	0	2	$2\sigma^4$
$\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, n)$	0	1	σ^4

5.5 *Approximate Bias, Variance, and Mean Squared Error of the Folded CvM Estimators*

Now we can give approximate values for the bias, variance, and the mean squared error of the folded CvM estimators considered. We base our computations on the expected values, variances, and covariance from Theorems 1.2.10 and 4.4.1 and the computations carried out in Examples 4.4.2–4.4.5. Table 6 synthesizes these results. It is important to notice that this table is valid for any stochastic process satisfying Assumptions A.1–A.4 and A.6. Among the considered weight functions, it is clear that for large values of m the estimator with smaller MSE is $\overline{C}_{0,1}(g_0, n)$, regardless of the underlying stochastic process. It is interesting to observe that, even though $\overline{C}_{0,1}(g_0, n)$ is not a first-order unbiased estimator (in contrast with some of the other CvM estimators) its variance is so much smaller that it compensates for its bias.

Our empirical examples in Section 5.2 show how $\overline{C}_{0,1}(g_0, n)$ converges more slowly to σ^2 than the rest of the estimators under consideration. This is due to the fact that $\overline{C}_{0,1}(g_0, n)$ is not a first-order unbiased estimator. However, it is the estimator with smallest standard error followed closely by the first-order unbiased estimator $\overline{C}_{0,1}(g_6^*, g_{(6)}^*, n)$. In Table 6 we can also see that $\overline{C}_{0,1}(g_6^*, g_{(6)}^*, b, m)$ has the second smallest variance (i.e., second smallest standard error) in consonance with the empirical examples.

5.6 Correction Factors

Given that the coefficient of σ^2 in the expected value of $A_1(w, n)$ in Theorem 3.2.3 is not equal to one, for us to say that the estimator is first-order unbiased is not quite correct. Not only do we require the coefficient of γ to be zero, but also the coefficient of σ^2 should be one. Fortunately, $W_n^* \neq 0$ for large enough n, which means that we can divide by it, improving the estimator, as pointed out on Remark 3.2.5.

In this section, we compute the coefficient of σ^2 in the expected value of $A_1(w_{\sin,1}, n)$ in Theorem 3.2.3. The reason we do that, is to provide the reader with the correction factors for different values of the sample size n to transform $A_1(w_{\sin,1}, n)$ into a first-order unbiased estimator for σ^2. In Table 7, we give the exact values of the coefficient of σ^2 in the expected value of $A_1(w_{\sin,1}, n)$ for n ranging from 2^1 to 2^8 under the $W_{D,n}^*$ header. We provide a closed formula for $W_{D,n}^*$ in the proof of Theorem 3.2.3 given by Equation (B.3.1), but that formula gets more complex as n increases; therefore, we only compute the exact values for powers of 2 up to 2^8. In fact, the number of terms needed to compute $W_{D,n}^*$ for the last

Table 6: Approximate Asymptotic Bias, Variance, and Mean Squared Error for Folded CvM Estimators of the Variance Parameter of a Stationary Stochastic Process.

Folded CvM Estimators	Approx. (n/γ)Bias	Approx. $(1/\sigma^4)$Var	Approx. MSE
$C_0(g_0, n)$	5	0.8	$\frac{25\gamma^2}{n^2} + 0.8\sigma^4$
$C_1(g_0, n)$	8	0.8	$\frac{64\gamma^2}{n^2} + 0.8\sigma^4$
$\overline{C}_{0,1}(g_0, n)$	6.5	0.5	$\frac{169\gamma^2}{4n^2} + 0.5\sigma^4$
$C_0(g_2^*, n)$	0	1.73	$1.73\sigma^4$
$C_1(g_{(2)}^*, n)$	0	2.06	$2.06\sigma^4$
$\overline{C}_{0,1}(g_2^*, g_{(2)}^*, n)$	0	1	σ^4
$C_0(g_4^*, n)$	0	1.04	$1.04\sigma^4$
$C_1(g_{(4)}^*, n)$	0	1.14	$1.14\sigma^4$
$\overline{C}_{0,1}(g_4^*, g_{(4)}^*, n)$	0	0.68	$0.68\sigma^4$
$C_0(g_6^*, n)$	0	0.81	$0.81\sigma^4$
$C_1(g_{(6)}^*, n)$	0	0.85	$0.85\sigma^4$
$\overline{C}_{0,1}(g_6^*, g_{(6)}^*, n)$	0	0.52	$0.52\sigma^4$

entry is close to 17,000.

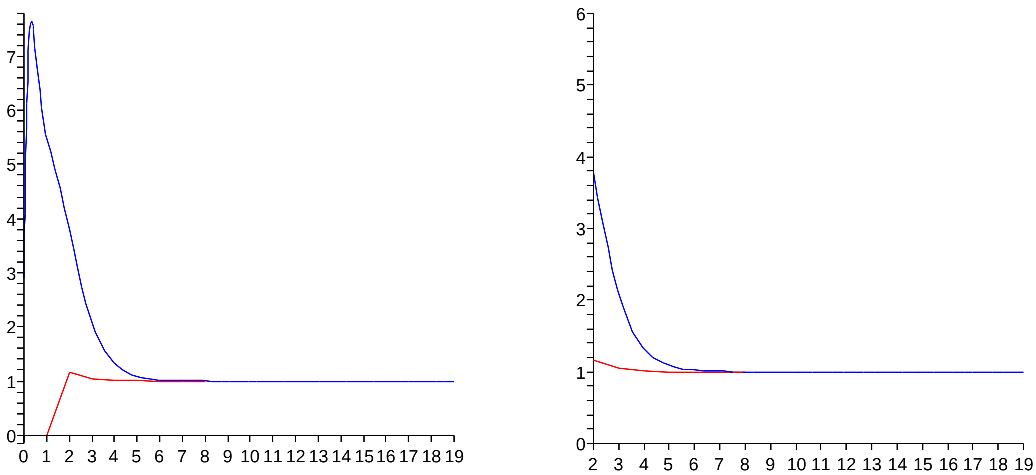
Fortunately, we have W_n^*, which allows us to have good approximations for $W_{D,n}^*$ for larger values of n. We recommend using the exact correction factors given in Table 7 up to $n = 2^8$, and to use the approximate correction factors for larger values of n if they are needed. Observe that these correction factors for large values of n are very close to one. This fact is due to the way we constructed $w_{\sin,1}$. In fact, W_n^* is asymptotically equal to one.

In Table 7, the entries marked † are too hard to compute exactly compared to the approximate value given by W_n^*. Note that the values of W_n^* for small n are quite bad (as a result of ignoring terms that *are* significant for small n); hence the recommendation is to use the $W_{D,n}^*$ column instead.

Table 7: Correction Factors for $w_{\sin,1}$.

$log_2(n)$	n	W_n^*	$W_{D,n}^*$
2	4	3.791529	1.166336
3	8	2.093500	1.042043
4	16	1.322485	1.010424
5	32	1.085632	1.002599
6	64	1.021924	1.000649
7	128	1.005537	1.000162
8	256	1.001391	1.000041
9	512	1.000348	†
10	1024	1.000087	†
11	2048	1.000022	†
12	4096	1.000005	†
13	8192	1.000001	†
14	16384	1.000000	†
15	32768	1.000000	†
16	65536	1.000000	†
17	131072	1.000000	†
18	262144	1.000000	†
19	524288	1.000000	†

In Figure 3 we show what the correction factors look like graphically. The horizontal axis has $\log_2(n)$, where n corresponds to the sample size used by the estimator. In blue we draw W_n^* (as a continuous function of n) and in red we draw $W_{D,n}^*$. Figure 3.b shows a detail of Figure 3.a, zooming into a more-interesting range of values.



a. Overview **b.** Detail

Figure 3: Correction Factors for $w_{\sin,1}$.
On the abscissas we have $\log_2(n)$, where n is the sample size. The red line graphs $W_{D,n}^*$ and the blue one graphs W_n^*.

CHAPTER VI

BATCHED FOLDED ESTIMATORS

All of our work so far has assumed that we have only one long batch of n observations. In actual applications, one could organize the data by breaking the n observations into b contiguous, nonoverlapping batches, each of size m in such a way that $n = bm$ as a standard variance reduction technique. In this chapter, we introduce the batched versions of the folded estimators studied in Chapters III and IV. In Section 6.1, we explain how to split a long simulation run into b batches of size m, and introduce the required notation. In Section 6.2, we define the level-k folded area estimator within each batch. Then, we average the b folded area estimators to produce the batched version of the folded area estimator. In Section 6.3, we repeat this procedure but this time we do it for the folded CvM estimator. Finally, we present the empirical results obtained via Monte Carlo simulation experiments in Section 6.4.

6.1 Introduction

Again, let $\{X_i,\, 1 \le i \le n\}$ be the output of a long simulation run from a stationary process with a finite variance parameter σ^2. Split the output data into b adjacent, nonoverlapping batches, each consisting of m observations, where with little loss of generality we assume that $n = mb$. Therefore, the i^{th} batch consists of the observations

$$X_{(i-1)m+1}, X_{(i-1)m+2}, \ldots, X_{im}$$

for $i = 1, 2, \ldots, b$, and the i^{th} batch mean is the sample average of the observations from this batch:

$$\overline{X}_{i,m} \equiv \frac{1}{m} \sum_{j=1}^{m} X_{(i-1)m+j}.$$

Definition 6.1.1 *The level-*0 *STS from the i^{th} batch is defined as follows:*

$$T^{(0)}_{i,m}(t) \equiv \frac{\lfloor mt \rfloor \left(\overline{X}_{i,m} - \overline{X}_{i,\lfloor mt \rfloor} \right)}{\sigma\sqrt{m}} \quad \textit{for } 0 \le t \le 1 \textit{ and } 1 \le i \le b, \tag{6.1.1}$$

where

$$\overline{X}_{i,j} \equiv \frac{1}{j}\sum_{k=1}^{j} X_{(i-1)m+k} \quad \textit{for } 1 \le j \le m \textit{ and } 1 \le i \le b.$$

The following definition is similar to Definition 2.3.3.

Definition 6.1.2 *The level-k STS from the i^{th} batch is*

$$T^{(k)}_{i,m}(t) \equiv T^{(k-1)}_{i,m}\left(\frac{t}{2}\right) - T^{(k-1)}_{i,m}\left(1-\frac{t}{2}\right) \quad \textit{for } k \ge 1,\ 1 \le i \le b,\ \textit{and } 0 \le t \le 1. \tag{6.1.2}$$

6.2 Batched Folded Area Estimator

Definition 6.2.1 *For each $k \ge 1$ and for all $1 \le i \le b$, the level-k folded weighted STS area estimator for σ^2 from the i^{th} batch is defined by*

$$A_{k,i}(w,m) \equiv N^2_{k,i}(w,m) \equiv \left[\frac{1}{m}\sum_{\ell=1}^{m} w\left(\frac{\ell}{m}\right)\sigma T^{(k)}_{i,m}\left(\frac{\ell}{m}\right)\right]^2, \tag{6.2.1}$$

where $T^{(k)}_{i,m}(\cdot)$ is given by Equation (6.1.2), and $w(\cdot)$ is a weight function satisfying Assumption A.5.

Definition 6.2.2 *For each $k \ge 1$, the level-k batched folded area estimator for σ^2 is defined by*

$$\overline{A}_k(w,b,m) \equiv \frac{1}{b}\sum_{i=1}^{b} A_{k,i}(w,m), \tag{6.2.2}$$

where $w(\cdot)$ is a weight function satisfying Assumption A.5.

The next theorem states a result concerning the limiting distribution of the batched folded area estimator $\overline{A}_k(w,b,m)$.

Theorem 6.2.3 *If Assumptions A hold, then*

$$\overline{A}_k(w,b,m) \xrightarrow{\mathcal{D}} \sigma^2\chi^2_b/b \quad \textit{as } m \to \infty. \tag{6.2.3}$$

Proof: Since the batch means computed from different nonoverlapping batches are asymptotically independent and jointly normal (see Fishman [17]) as their pairwise covariances converge to zero as $m \to \infty$, for fixed b, the random variables $\{T^{(k)}_{i,m}(\cdot), i = 1, 2, \ldots, b\}$ are also asymptotically independent as functions of these batch means. Moreover, $\{T^{(k)}_{i,m}(\cdot), i =$

$1, 2, \ldots, b\}$ converge to Brownian bridge processes as m becomes large. This fact is a direct consequence of Lemma A.1.5 in Appendix A. As a result, we can conclude that the corresponding $\{A_{k,i}(w, m), i = 1, 2, \ldots, b\}$ are asymptotically independent as $m \to \infty$, and by Theorem 3.2.1 each $A_{k,i}(w, m) \xrightarrow{\mathcal{D}} \sigma^2 \chi_1^2$, so that (6.2.3) holds. $\Box$

Naturally, we are ready to state now the analogous for the batched folded area estimator of Theorem 3.2.3 as follows:

Theorem 6.2.4 *Suppose $\{X_i, i \geq 1\}$ is a stochastic process satisfying Assumptions A.1–A.5. Also assume that the family of random variables $\{A_{1,i}^2(w, m),\ m = 1, 2, \ldots\}$ is uniformly integrable for each i, $1 \leq i \leq b$. Let*

$$\begin{aligned} W_m^* = {} & W^2 - \overline{W}^2 + \frac{w(0)}{m}(2\overline{W} - W) \\ & - 2\int_0^{1/2} u \check{w}_m(u) \check{W}_m(u)\, du - 2\int_0^{1/2} u \widehat{w}_m(u) \widehat{W}_m(u)\, du \\ & + \frac{\check{w}_m(0)}{m} \int_0^{1/2} u \check{w}_m(u)\, du, \end{aligned} \tag{6.2.4}$$

where $W(\cdot)$ and $\overline{W}$ are defined in Theorem 1.2.5 and for $m \geq 1$,

$$\widehat{w}_m(t) \equiv w(2t) + w\left(2t + 1/m\right) \quad \textit{for } 0 \leq t \leq 1/2 \tag{6.2.5}$$

$$\widehat{W}_m(t) \equiv \int_0^t \widehat{w}_m(s)\, ds \quad \textit{for } 0 \leq t \leq 1/2 \tag{6.2.6}$$

$$\check{w}_m(t) \equiv w(2t) + w\left(2t - 1/m\right) \quad \textit{for } 0 \leq t \leq 1/2 \tag{6.2.7}$$

$$\check{W}_m(t) \equiv \int_0^t \check{w}_m(s)\, ds \quad \textit{for } 0 \leq t \leq 1/2. \tag{6.2.8}$$

Then,

$$\mathrm{E}(\overline{A}_1(w, b, m)) = \mathrm{E}(A_{1,i}(w, m)) = W_m^* \sigma^2 + \frac{\overline{W}^2 \gamma}{m} + \mathrm{o}(1/m) \tag{6.2.9}$$

and for fixed b,

$$\mathrm{Var}(\overline{A}_1(w, b, m)) \approx \frac{1}{b}\mathrm{Var}(A_{1,i}(w, m)) \to \frac{1}{b}\mathrm{Var}(A_1(w)) = \frac{1}{b}\mathrm{Var}(\sigma^2 \chi_1^2) = \frac{2\sigma^4}{b} \quad \textit{as } m \to \infty. \tag{6.2.10}$$

Proof: It is a direct consequence of Theorem 3.2.3, and the fact that as m becomes large (with fixed b), the $T_{i,m}^{(1)}$, $1 \leq i \leq b$, converge to Brownian bridge processes (since the

Brownian motion has independent increments). Indeed, since the batch means are jointly normal and asymptotically independent (see Fishman [17]) as their pairwise covariances converge to zero when $m \to \infty$, for fixed b, the random variables $\{T_{i,m}^{(k)}(\cdot), i = 1, 2, \ldots, b\}$ are also asymptotically independent as functions of these batch means. Moreover, $\{T_{i,m}^{(k)}(\cdot), i = 1, 2, \ldots, b\}$ converge to Brownian bridge processes as m becomes large. This fact is a direct consequence of Lemma A.1.5 in Appendix A. In that case, the $A_{1,i}(w, m)$'s for $1 \le i \le b$ are approximately independent, and so as $m \to \infty$, Equation (6.2.10) follows. $\square$

The following theorem is a direct consequence of Theorem 3.4.1, and the asymptotic independence of the batch means.

Theorem 6.2.5 *Let $f_i(\cdot)$ be a function satisfying Assumption A.5 for $i = 0, 1$. For $i = 0, 1$, let $F_i(s) \equiv \int_0^s f_i(u)\,du$ for $0 \le s \le 1$, and $\overline{F}_i(t) \equiv \int_0^t F_i(s)\,ds$ for $0 \le t \le 1$. Then, under Assumptions A.1–A.4,*

$$\begin{aligned} &\operatorname{Cov}(\overline{A}_0(f_0, b, m), \overline{A}_1(f_1, b, m)) \\ &\quad \approx \frac{2\sigma^4}{b}\left[\int_0^1 f_1(s)\left\{\overline{F}_0\left(1 - \frac{s}{2}\right) - \overline{F}_0\left(\frac{s}{2}\right)\right\} ds - \overline{F}_0(1)\overline{F}_1(1)\right]^2 \end{aligned}$$

for large values of m and fixed b.

Proof: First, observe that for large values of m and fixed b,

$$\operatorname{Cov}(\overline{A}_{k,i}(f_0, b, m), \overline{A}_{\ell,j}(f_1, b, m)) \approx 0 \quad \text{for every } i \ne j \text{ for all levels } k \text{ and } \ell \tag{6.2.11}$$

as a consequence of the asymptotic independence of the batch means. Now,

$$\begin{aligned} \operatorname{Cov}(\overline{A}_0(f, b, m), \overline{A}_1(f, b, m)) &\equiv \operatorname{Cov}(\frac{1}{b}\sum_{i=1}^{b} A_{0,i}(f_0, m), \frac{1}{b}\sum_{j=1}^{b} A_{1,j}(f_1, m)) \\ &= \frac{1}{b^2}\sum_{i=1}^{b}\sum_{j=1}^{b} \operatorname{Cov}(A_{0,i}(f_0, m), A_{1,j}(f_1, m)) \\ &\approx \frac{1}{b^2}\sum_{i=1}^{b} \operatorname{Cov}(A_{0,i}(f_0, m), A_{1,i}(f_1, m)) \\ &\qquad \text{(because of Equation (6.2.11))} \\ &= \frac{2\sigma^4}{b}\left[\int_0^1 f_1(s)\left\{\overline{F}_0\left(1 - \frac{s}{2}\right) - \overline{F}_0\left(\frac{s}{2}\right)\right\} ds - \overline{F}_0(1)\overline{F}_1(1)\right]^2 \end{aligned}$$

by Theorem 3.4.1. $\square$

Remark 6.2.6 If we take $f_0(t) = f_1(t) = w_0(t) = \sqrt{12}$, then

$$\int_0^1 f_1(s) \left\{ \overline{F}_0 \left(1 - \frac{s}{2}\right) - \overline{F}_0 \left(\frac{s}{2}\right) \right\} ds - \overline{F}_0(1)\overline{F}_1(1) = 0. \tag{6.2.12}$$

Moreover, (6.2.12) is zero as well if we take $f_0(t) = w_{\cos,k}(t)$ for any $k \geq 1$, and for any choice of $f_1(t)$. This is an extremely pleasant surprise since it will allow us to compute the approximate variance of the averaged batched folded area estimators by dividing the sum of the variances of the individual batched folded area estimators divided by $4b$ whenever the weight functions are w_0, or $w_{\cos,1}$ and $w_{\sin,1}$. We review these results in Table 8. $\triangleleft$

6.3 Batched Folded CvM Estimator

Definition 6.3.1 *For each $k \geq 1$ and for all $1 \leq i \leq b$, the level-k folded weighted CvM estimator for σ^2 from the i^{th} batch is defined by*

$$C_{k,i}(g, m) \equiv \frac{1}{m} \sum_{j=1}^{m} g\left(\frac{j}{m}\right) \left[\sigma T_{i,m}^{(k)}\left(\frac{j}{m}\right)\right]^2,$$

where $T_{i,m}^{(k)}(\cdot)$ is given by Definition 6.1.2, and $g(\cdot)$ is a weight function satisfying Assumption A.6.

Definition 6.3.2 *For each $k \geq 1$, the level-k batched folded CvM estimator for σ^2 is defined by*

$$\overline{C}_k(g, b, m) \equiv \frac{1}{b} \sum_{i=1}^{b} C_{k,i}(g, m) \tag{6.3.1}$$

where $g(\cdot)$ is a weight function satisfying Assumption A.6.

The next theorem states a result concerning the limiting distribution of the batched folded CvM estimator $\overline{C}_k(w, b, m)$.

Theorem 6.3.3 *If Assumptions A hold, then*

$$\overline{C}_k(g, b, m) \xrightarrow{\mathcal{D}} \frac{1}{b} \sum_{i=1}^{b} C_{k,i}(g) \quad \text{as } m \to \infty \tag{6.3.2}$$

where the $\{C_{k,i}(g) : i = 1, \ldots, b\}$ are i.i.d. random variables each having the distribution of $C_k(g)$.

Proof: Since the batch means are jointly normal and asymptotically independent (see Fishman [17]) as their pairwise covariances converge to zero as $m \to \infty$, for fixed b, the random variables $\{T_{i,m}^{(k)}(\cdot), i = 1, 2, \ldots, b\}$ are also asymptotically independent as functions of these batch means. Moreover, $\{T_{i,m}^{(k)}(\cdot),\ i = 1, 2, \ldots, b\}$ converge to Brownian bridge processes as m becomes large. This fact is a direct consequence of Lemma A.1.5 in Appendix A. Since the $\{T_{i,m}^{(k)}(\cdot),\ i = 1, 2, \ldots, b\}$ converge to Brownian bridge processes as m becomes large, we can conclude that the corresponding $\{C_{k,i}(g, m),\ i = 1, 2, \ldots, b\}$ are asymptotically independent as $m \to \infty$, and by Theorem 4.1.3, $C_{k,i}(g, m) \xrightarrow{\mathcal{D}} C_{k,i}(g)$ for each $i = 1, 2, \ldots, b$. Therefore, we have that (6.3.2) holds. $\square$

The following result is a direct consequence of Theorem 4.2.1 and the comments made in the proof of Theorem 6.2.4.

Theorem 6.3.4 *Suppose $\{X_i, i \geq 1\}$ is a stochastic process satisfying Assumptions A.1–A.4, and let $g(\cdot)$ be a weight function satisfying Assumption A.6. Further, suppose that the family of random variables $\{C_{1,i}^2(g, m) : m = 1, 2, \ldots\}$ is uniformly integrable for each i $1 \leq i \leq b$,*

$$G^* \equiv \int_0^1 \int_0^t g(s)\, ds dt - \int_0^1 (1-t)^2 g(t)\, dt,$$

and

$$I \equiv \int_0^1 (t^2 - 2t + 2) g(t)\ dt.$$

Then

$$\mathrm{E}(\overline{C}_1(g, b, m)) = \mathrm{E}(C_{1,i}(g, m)) = G^* \sigma^2 + \frac{I\gamma}{m} + \mathrm{o}(1/m),$$

and for fixed b,

$$\begin{aligned} \mathrm{Var}(\overline{C}_1(g, b, m)) = \frac{1}{b}\mathrm{Var}(C_{1,i}(g, m)) &\to \frac{1}{b}\mathrm{Var}(C_1(g)) \\ &= \frac{4\sigma^4}{b} \int_0^1 g(t)(1-t)^2 \int_0^t g(s) s^2\ ds\ dt \quad \textit{as } m \to \infty. \end{aligned}$$

Theorem 6.3.5 *Let $f_i(\cdot)$ be a function satisfying Assumption A.6 for $i = 0, 1$. Then, for large m and fixed b,*

$$\begin{aligned}\mathrm{Cov}(\overline{C}_0(f_0,b,m),\overline{C}_1(f_1,b,m)) \approx\ & \frac{2\sigma^4}{b}\int_0^1\int_0^{s/2} f_0(t)f_1(s)t^2(1-s)^2\,dt\,ds \\ &+ \frac{2\sigma^4}{b}\int_0^1\int_{s/2}^{1-s/2} f_0(t)f_1(s)s^2(1/2-t)^2\,dt\,ds \\ &+ \frac{2\sigma^4}{b}\int_0^1\int_{1-s/2}^{1} f_0(t)f_1(s)(1-s)^2(1-t)^2\,dt\,ds.\end{aligned}$$

Proof: Similar to the proof of Theorem 6.2.11, but this time we use Theorem 4.4.1 instead of Theorem 3.4.1. □

Remark 6.3.6 Note that Theorem 6.3.5 allows us to recycle all the computations done in Examples 4.4.2–4.4.5 when computing the variance of the averaged batched folded CvM estimators. We just need to take the variance of the averaged weighted folded CvM estimators and divide it by b. We review all these computations on Table 8.

6.4 Empirical Results

We present empirical examples illustrating the effect of batching in the behavior of the following estimators for σ^2:

- $\overline{A}_{0,1}(w_0,b,m) \equiv (\overline{A}_0(w_0,b,m) + \overline{A}_1(w_0,b,m))/2$.
- $\overline{A}_{0,1}(w_{\cos,1},w_{\sin,1},b,m) \equiv (\overline{A}_0(w_{\cos,1},b,m) + \overline{A}_1(w_{\sin,1},b,m))/2$.
- $\overline{C}_{0,1}(g_0,b,m) \equiv (\overline{C}_0(g_0,b,m) + \overline{C}_1(g_0,b,m))/2$.
- $\overline{C}_{0,1}(g_2^*,g_{(2)}^*,b,m) \equiv (\overline{C}_0(g_2^*,b,m) + \overline{C}_1(g_{(2)}^*,b,m))/2$.
- $\overline{C}_{0,1}(g_4^*,g_{(4)}^*,b,m) \equiv (\overline{C}_0(g_4^*,b,m) + \overline{C}_1(g_{(4)}^*,b,m))/2$.
- $\overline{C}_{0,1}(g_6^*,g_{(6)}^*,b,m) \equiv (\overline{C}_0(g_6^*,b,m) + \overline{C}_1(g_{(6)}^*,b,m))/2$.

Note that if $b = 1$ and $m = n$ we have the estimators from the previous chapters. For instance, $\overline{C}_1(g_{(6)}^*,1,n) = C_1(g_{(6)}^*,n)$ and $\overline{C}_{0,1}(g_6^*,g_{(6)}^*,1,n) = \overline{C}_{0,1}(g_6^*,g_{(6)}^*,n)$.

The reason we are only considering the average of batched estimators is that their performance in Chapter V based on a single batch of size n was shown to be better than that of their counterparts. These examples involve Monte Carlo simulations of the AR(1) process described in Chapter V with $\varphi = 0.9$.

For each figure, we have the following:

- The horizontal axis corresponds to $\log_2(n)$. We use all powers of two between 2^9 and 2^{19} for the sample sizes.

- The vertical axis corresponds to the sample average of the respective estimator or the sample standard error of the estimator (sample average in the 'a' sub-figure and sample standard error in the 'b' sub-figure).

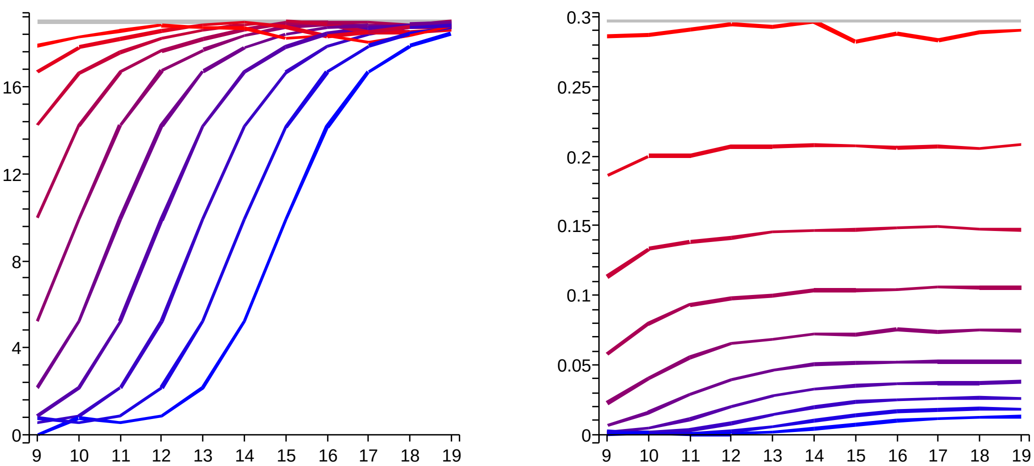
- There were $2^{12} = 4096$ independent replications of each estimator (and for each sample size, batch number combination).

- The data points were generated with a program written in C and the results were graphed using Maple.

- The colors indicate the batch counts (numbers of batches), ranging from 2^0 (red) to 2^9 (blue), and each consecutive color indicates a factor of two on the batch count. Those extreme values for the batch counts were computed to get a clearer picture of the situation. In practice we would use batch counts around 32 (shown in purple in the figures).

- The line in grey is a reference value. In the 'a' sub-figure, this is the theoretical expected value of the estimators for $\sigma^2 = 19$. For the 'b' sub-figure, it is the theoretical standard error for $2^0 = 1$ batch for an area estimator.

The common theme in Figures 4 to 9 is that as the number of batches increases, the convergence to σ^2 of the estimators is slower. In contrast, we observe that the sample standard error decreases as the number of batches grows. This comes as no surprise since batching is a widely-used variance reduction technique. Also of note is the previously observed similarity in the behavior of the area estimators and the degree-2 polynomial weight CvM (see Figures 4, 5, and 7). As in the case with one batch, the CvM estimators of higher-degree polynomial weight have sample standard errors decreasing towards the sample standard error of the constant weight CvM estimator (see Figures 6 to 9).



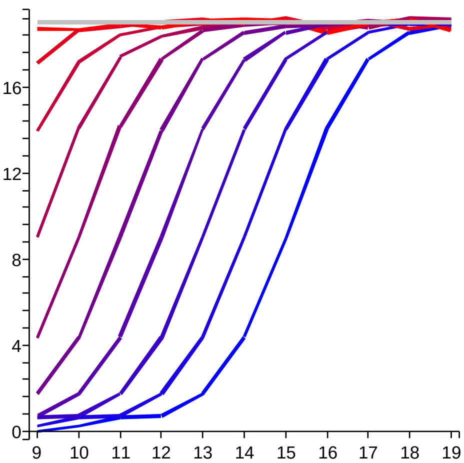
a. Sample Average **b.** Sample Standard Error

Figure 4: Batch Behavior of $\overline{A}_{0,1}(w_0, b, m)$.
On the abscissas we have $\log_2(n)$, where n is the sample size. The colors indicate the batch counts, ranging from 1 (red) to 512 (blue), and each consecutive color indicates a factor of two on the batch count. The line in grey is a reference value.

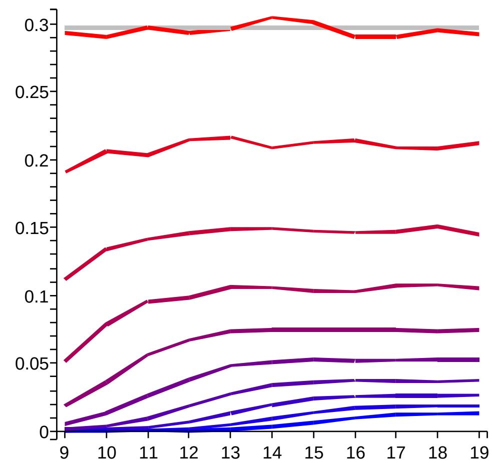
Finally, we compare all the averaged estimators for a given number of batches. As before, in the horizontal axis we have $\log_2(n)$, the area estimators are in warm hues and the CvM estimators are in cold hues, that is:

Red:	$\overline{A}_{0,1}(w_0, b, m)$
Orange:	$\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, b, m)$
Black:	$\overline{C}_{0,1}(g_0, b, m)$
Blue:	$\overline{C}_{0,1}(g_2^*, g_{(2)}^*, b, m)$
Greenish Blue:	$\overline{C}_{0,1}(g_4^*, g_{(4)}^*, b, m)$
Bluish Green:	$\overline{C}_{0,1}(g_6^*, g_{(6)}^*, b, m)$

In grey is the reference value, $\sigma^2 = 19$ (for the 'a' sub-figure) or the theoretical standard error for the area estimator (for the 'b' sub-figure).

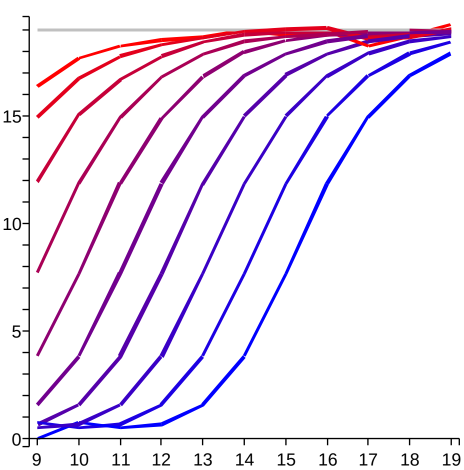


a. Sample Average

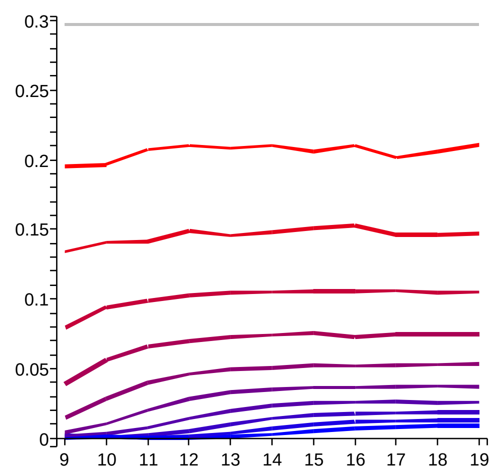


b. Sample Standard Error

Figure 5: Batch Behavior of $\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, b, m)$.
On the abscissas we have $\log_2(n)$, where n is the sample size. The colors indicate the batch counts, ranging from 1 (red) to 512 (blue), and each consecutive color indicates a factor of two on the batch count. The line in grey is a reference value.

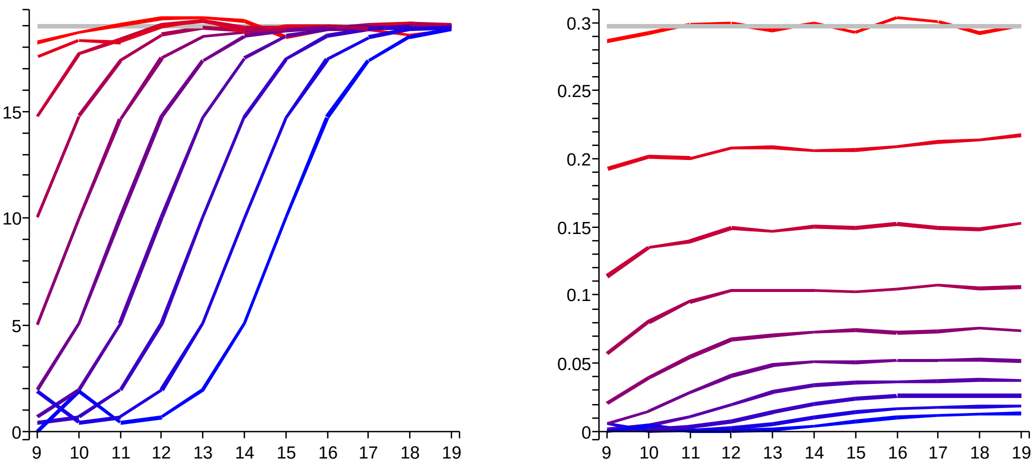


a. Sample Average



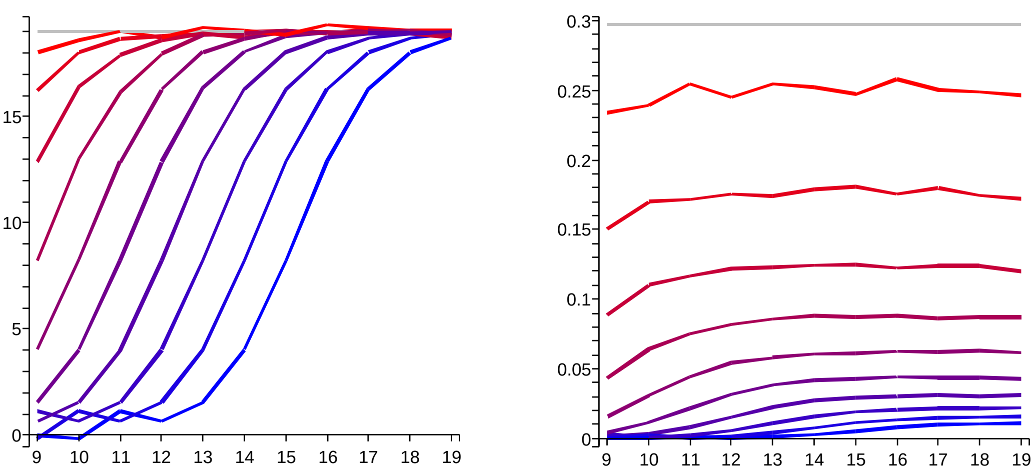
b. Sample Standard Error

Figure 6: Batch Behavior of $\overline{C}_{0,1}(g_0, b, m)$.
On the abscissas we have $\log_2(n)$, where n is the sample size. The colors indicate the batch counts, ranging from 1 (red) to 512 (blue), and each consecutive color indicates a factor of two on the batch count. The line in grey is a reference value.



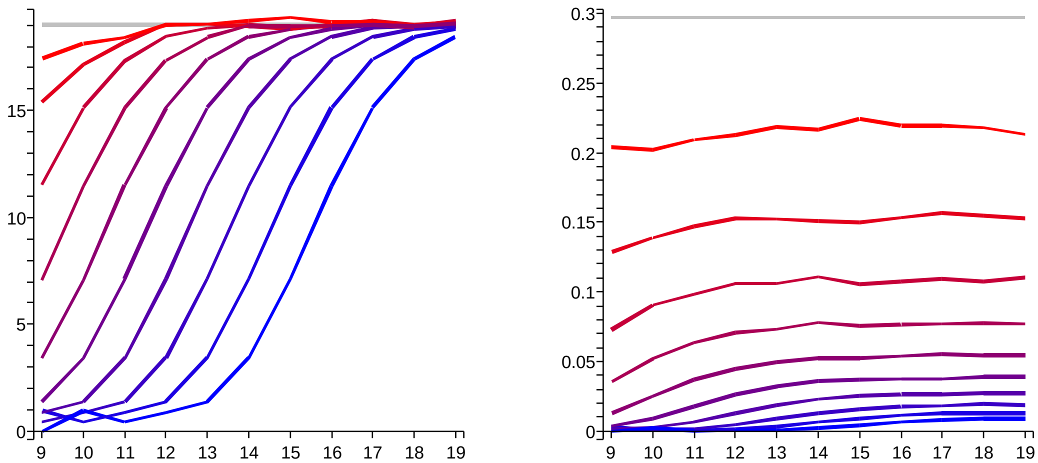
a. Sample Average **b.** Sample Standard Error

Figure 7: Batch Behavior of $\overline{C}_{0,1}(g_2^*, g_{(2)}^*, b, m)$.
On the abscissas we have $\log_2(n)$, where n is the sample size. The colors indicate the batch counts, ranging from 1 (red) to 512 (blue), and each consecutive color indicates a factor of two on the batch count. The line in grey is a reference value.



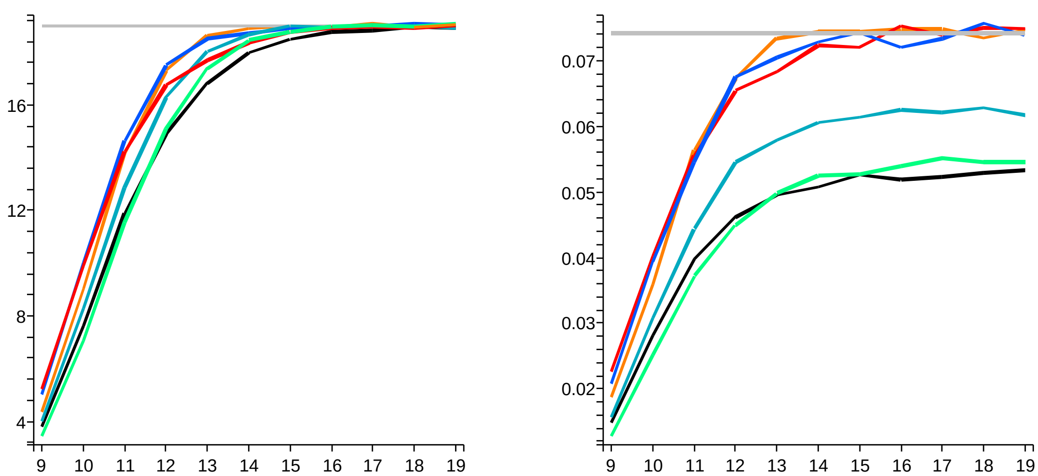
a. Sample Average **b.** Sample Standard Error

Figure 8: Batch Behavior of $\overline{C}_{0,1}(g_4^*, g_{(4)}^*, b, m)$.
On the abscissas we have $\log_2(n)$, where n is the sample size. The colors indicate the batch counts, ranging from 1 (red) to 512 (blue), and each consecutive color indicates a factor of two on the batch count. The line in grey is a reference value.



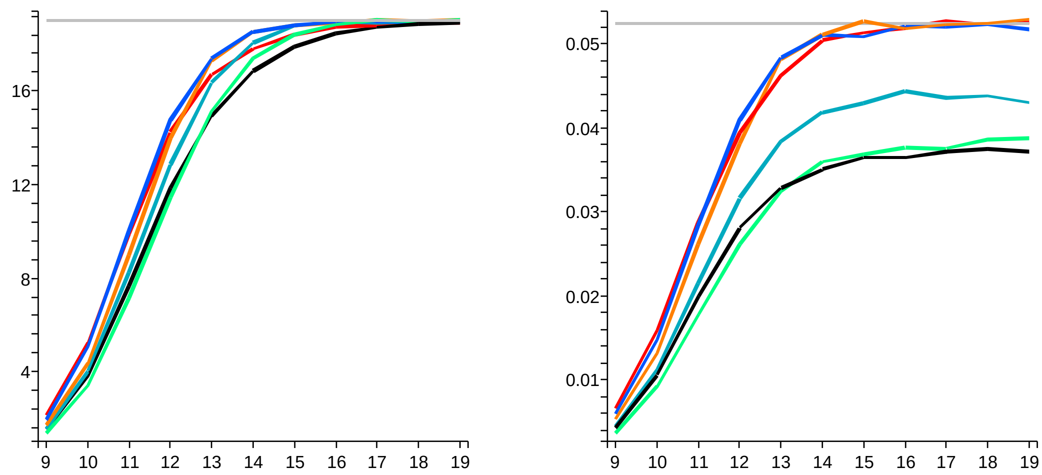
a. Sample Average　　　　**b.** Sample Standard Error

Figure 9: Batch Behavior of $\overline{C}_{0,1}(g_6^*, g_{(6)}^*, b, m)$.
On the abscissas we have $\log_2(n)$, where n is the sample size. The colors indicate the batch counts, ranging from 1 (red) to 512 (blue), and each consecutive color indicates a factor of two on the batch count. The line in grey is a reference value.



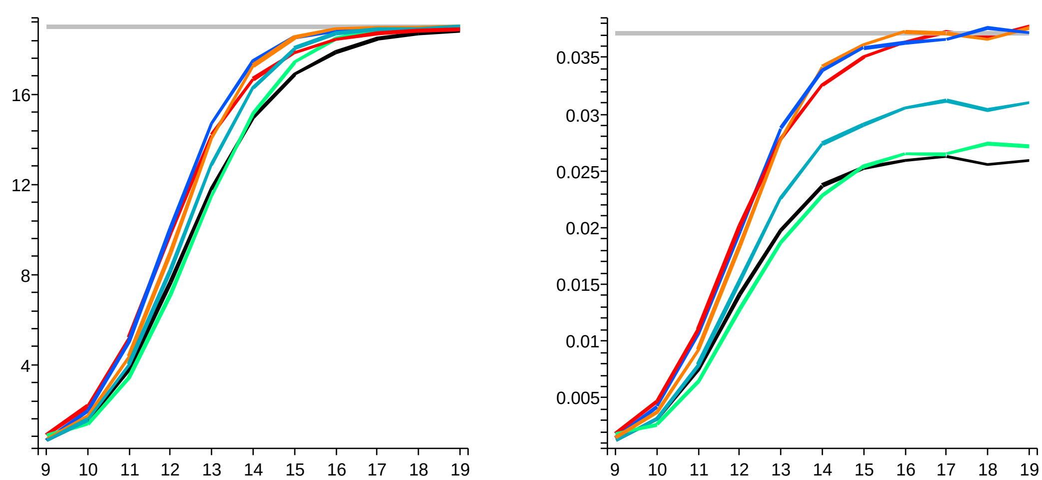
a. Sample Average　　　　**b.** Sample Standard Error

Figure 10: Comparison of Estimators Using 16 batches.
On the abscissas we have $\log_2(n)$, where n is the sample size. The area estimators are in warm hues and the CvM estimators are in cold hues. The line in grey is a reference value.



a. Sample Average **b.** Sample Standard Error

Figure 11: Comparison of Estimators Using 32 batches.
On the abscissas we have $\log_2(n)$, where n is the sample size. The area estimators are in warm hues and the CvM estimators are in cold hues. The line in grey is a reference value.



a. Sample Average **b.** Sample Standard Error

Figure 12: Comparison of Estimators Using 64 batches.
On the abscissas we have $\log_2(n)$, where n is the sample size. The area estimators are in warm hues and the CvM estimators are in cold hues. The line in grey is a reference value.

As before, we observe that, for fixed sample size, increasing the number of batches slows down the convergence, not only to σ^2 but also to the theoretical standard error; this suggests that there is no need to go overboard on the batch number.

The two estimators that converge the fastest to σ^2 seem to be $\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, b, m)$ and $\overline{C}_{0,1}(g_2^*, g_{(2)}^*, b, m)$, but they also have the largest sample standard errors (this is consistent with the theoretical standard error for an averaged area estimator). Also, the constant weight CvM seems to have the smallest sample standard error.

Note that the area estimators and the degree-two CvM estimator are very close to each other in all the graphs (an observation we promised to refine at the end of Chapter V). Recall that $\overline{A}_{0,1}(w_0, b, m)$ is not first-order unbiased while the other two are.

Note also that the sample average and sample standard error of the higher-degree CvM estimators seem to be "converging" to the constant weight CvM estimator's sample average and sample standard error (not shown here, but also computed is the degree-eight CvM estimator, which was almost indistinguishable from the constant weight CvM estimator). It is interesting that the constant weight CvM estimator is not first-order unbiased, while the other ones are. The constant weight CvM estimator has the slowest convergence while it has the smallest sample standard error.

Finally, the behavior seen in Figures 10 to 12 is also seen for larger number of batches.

6.5 Approximate Bias, Variance, and Mean Squared Error of the Averaged Batched Folded Estimators

At this point we give approximate values for the bias, variance, and the mean square error of all the averaged batched folded estimators we have considered. Table 8 can be filled thanks to Theorems 1.2.5, 1.2.10, 6.2.4, 6.2.5, 6.3.4, and 6.3.5, and the computations carried out in Examples 4.4.2–4.4.5. As before, it is important to notice that this table is valid for any stochastic process satisfying Assumptions A.1–A.4. For the considered weight functions, it is clear that for large values of m, the estimator with smallest MSE is $\overline{C}_{0,1}(g_0, b, m)$ independently of the underlying stochastic process. This is not completely unexpected in view of the results obtained in Section 5.5 and summarized in Table 6.

The empirical examples from Section 6.4 show how $\overline{C}_{0,1}(g_0, b, m)$ converges more slowly to σ^2 than the rest of the estimators under consideration. This is due to the fact that $\overline{C}_{0,1}(g_0, b, m)$ is not a first-order unbiased estimator. However, from the same empirical examples, we see that $\overline{C}_{0,1}(g_0, b, m)$ has a variance that is smaller than those of the other estimators, whether they are first-order unbiased or not.

The first-order unbiased estimator that most closely matches the performance of $\overline{C}_{0,1}(g_0, b, m)$ is $\overline{C}_{0,1}(g_6^*, g_{(6)}^*, b, m)$—it converges at about the same rate and has a small standard error. All this corroborates the information in Table 8.

Notice that Table 8 makes it clear that the behavior of the estimators $\overline{A}_{0,1}(w_0, b, m)$, $\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, b, m)$ and $\overline{C}_{0,1}(g_2^*, g_{(2)}^*, b, m)$ observed in Sections 5.3 and 6.4 is not fortuitous; the three estimators have asymptotically similar bias and variance and thus asymptotically similar mean squared errors—this is the subtle insight promised in Section 5.2 of Chapter V.

Table 8: Approximate Asymptotic Bias, Variance, and Mean Squared Error for Batched Folded Estimators of the Variance Parameter of a Stationary Stochastic Process.

Averaged Batched Folded Estimators	Approx. (m/γ)Bias	Approx. (b/σ^4)Var	Approx. MSE
$\overline{A}_{0,1}(w_0, b, m)$	3	1	$\frac{9\gamma^2}{m^2} + \frac{\sigma^4}{b}$
$\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, b, m)$	0	1	$\frac{\sigma^4}{b}$
$\overline{C}_{0,1}(g_0, b, m)$	$\frac{13}{2}$	0.5	$\frac{169\gamma^2}{4m^2} + \frac{0.5\sigma^4}{b}$
$\overline{C}_{0,1}(g_2^*, g_{(2)}^*, b, m)$	0	1	$\frac{\sigma^4}{b}$
$\overline{C}_{0,1}(g_4^*, g_{(4)}^*, b, m)$	0	0.68	$\frac{0.68\sigma^4}{b}$
$\overline{C}_{0,1}(g_6^*, g_{(6)}^*, b, m)$	0	0.52	$\frac{0.52\sigma^4}{b}$

CHAPTER VII

CONCLUSIONS AND EXTENSIONS

7.1 Review

In this thesis, we introduced a new class of estimators for the variance parameter σ^2 of a stationary stochastic process $\{X_i, i \geq 1\}$. These new estimators are generalizations of the standardized time series weighted area and weighted Cramér-von Mises (CvM) estimators. The area and CvM estimators are based on the fact that a standardized time series converges to a Brownian bridge process. This thesis developed a new form of standardized time series that converges to a so-called level-i (or folded) Brownian bridge process.

Before proposing our new estimators for σ^2, we defined and then obtained a number of results on the folded Brownian bridges. In particular, we expressed the folded bridges in terms of the underlying Brownian bridge and Brownian motion processes. We then derived several interesting properties of various functionals of the folded bridges, e.g., the square of the weighted area under a bridge, and the area under the squared bridge—which correspond to the limiting versions of the folded area and CvM estimators, respectively. An important finding is that the areas under certain bridges from different levels are actually independent normal random variables.

At that point, we showed how to redefine the standardized time series so that it converges to the appropriate folded Brownian bridge. We were finally in the position to propose our new folded standardized time series weighted area and CvM estimators for σ^2. We showed that these estimators are asymptotically unbiased for σ^2 and have variance properties that are comparable to their "unfolded" counterparts. Thanks to a tedious analysis (relegated to the appendices), we even found weight functions that yield first-order unbiased versions of the estimators. We also demonstrated how to combine estimators from different levels, resulting in estimators having lower variance, while maintaining the same low bias.

In order to show that the estimators perform as advertised, we conducted analytical and Monte Carlo evaluations. First of all, we obtained exact results on the expected value and variance of the estimators when applied to a simple moving average process. We also undertook an empirical study involving AR(1) and M/M/1 waiting time processes. There were no surprises. The level-1 estimators performed about the same as their unfolded level-0 competitors; and linear combinations of the corresponding level-0 and level-1 estimators outperformed the individual estimators—as anticipated by the theory. Although we are loathe to declare a "winner" among the various versions of the estimators under consideration, we point out that the first-order unbiased linear combination estimators $\overline{A}_{0,1}(w_{\cos,1}, w_{\sin,1}, n)$ and $\overline{C}_{0,1}(g_2^\star, g_{(2)}^\star, n)$ always seemed to converge to the right expected value quickly while performing comparatively well in terms of variance.

7.2 Other Topics of Interest

We have a number of interesting problems that are the subjects of ongoing research. In what follows, we outline these problems and possible methods of attack.

1. *Higher Levels.* Our detailed theoretical and empirical analyses primarily concerned level-1 folded estimators. What happens when we go to higher levels? Although we derived certain asymptotic properties of the expected value and variance of higher levels of the area and CvM estimators, we did not perform a careful analysis based on the mean-squared error of estimators from those levels, nor did we carry out any substantive Monte Carlo analysis. One question worth asking is "For a given total sample size n, how many levels of folding can we apply before the asymptotic approximations completely break down for practical purposes?" (A similar question is addressed to some extent in Foley and Goldsman [20] with respect to the number of orthogonal weights that an estimator could accommodate in practical situations.)

2. *Linear Combinations of Estimators.* Related to the above, we also intend to study the properties of different linear combinations of area and CvM estimators between and within higher levels. For instance, can we obtain orthogonal estimators within a level à la Foley and Goldsman [20]? And what about linear combinations of like estimators

between levels à la Corollary 2.2.3? Estimators constructed with these ideas in mind will likely have comparatively lower variance than their constituents.

3. *Confidence Intervals.* The work in this thesis has concerned the development and analysis of point estimators for σ^2. A natural use of such point estimators is to incorporate them in confidence intervals for the more-fundamental parameter μ_X—the mean of the stationary stochastic process under consideration. We believe that it will be straightforward to derive t-distribution-based confidence intervals for μ_X based on the usual pivot obtained by dividing the standardized sample mean by the square root of a folded area estimator—since a standard normal random variable over the square root of an independent χ^2 random variable has the t distribution.

   Similarly, we can derive confidence intervals for the variance parameter of the underlying stochastic process. Since the folded area estimators have a χ_1^2 limiting distribution, this task is easy even though we have only one degree of freedom. At this point, the batched folded area estimators come to our rescue by giving us for "free" b degrees of freedom, given that their limiting distribution is χ_b^2. Unfortunately, this exercise is not so easy when the variance estimator is a CvM random variable. In that case, we might use Monte Carlo methods to obtain the necessary quantiles of the associated confidence interval pivot (as in e.g., Goldsman, et al. [24]), or we might simply approximate the necessary distribution via moment matching methods. In fact, we can use this methodology for each of the weights proven to produce first-order unbiased estimators with relatively small variance. We can also perform goodness-of-fit tests for each weight function to try to make an educated guess about the limiting distributions of the associated estimators.

4. *Comprehensive Experimental Performance Evaluation.* What about a comprehensive experimental performance evaluation on the outputs of more complex discrete-event stochastic systems? After all, the AR(1) process and the M/M/1 queue waiting time process for 80% traffic intensity do not cover the full spectrum of output processes encountered in practice. We could develop a testbed of problems that can be used to

compare the performance of the folded variance estimators with that of the benchmark NBM and OBM variance estimators. All of these estimators should be tested out over a variety of problems of realistic complexity before they can be used for actual problems. In particular, this set of experiments could help us answer the following question: What are the limitations and drawbacks of the methodolgies proposed in this research? It would be premature to answer this question based only on the information drawn from our current experiments.

5. *Overlapping Techniques.* We can formulate the overlapping folded area and CvM estimators by splitting a long simulation run of size n into a number overlapped batches, computing within each batch the usual folded estimators, and averaging the folded estimators computed from each batch. This technique has proven to be very effective for variance reduction, as can be seen in Alexopoulos et al. [1]. It is a natural extension of our research to apply overlapping batching techniques to our folded estimators.

6. *Development of Software.* What about developing public domain software for computing folded variance estimators from simulation generated output processes? Our C source code could easily be adapted into a package to expand the functionality of a statistics system such as R—which provides a language and environment for statistical computing.

7. *Quality Control.* Can we apply these estimation techniques to calibrate the control limits of quality control charts? In manufacturing, these charts are usually created to detect when a production process has gone out of control. In practice, batch production processes can be highly correlated, and the limits of the control charts depend upon having good estimates for the variance parameter of the underlying stochastic process.

APPENDIX A

DISTRIBUTIONS

A.1 Preliminaries

To establish rigorously the validity of Theorem 3.2.1, we must first prove almost-sure continuity of the folding operation $\Psi : Y \in D[0,1] \to \Psi_Y \in D[0,1]$ defined by

$$\text{If } Y \in D[0,1], \text{ then we take } \Psi_Y(t) \equiv Y(t/2) - Y(1 - t/2) \text{ for every } t \in [0,1]. \tag{A.1.1}$$

To make this discussion self-contained, we introduce some standard symbolism and definitions.

Definition A.1.1 *Let $\mathcal{W}(\cdot)$ denote a standard Brownian motion process on $[0,\infty)$ so that $\mathrm{E}(\mathcal{W}(t)) = 0$ and $\mathrm{Var}(\mathcal{W}(t)) = t$ for all $t \geq 0$. Later in the discussion, we will also let $\mathcal{W}(t)$ (for $t \in [0,1]$) denote a specific realization of this process that has been arbitrarily selected from a suitable probability-one subspace of $D[0,1]$.*

Definition A.1.2 *Let Λ denote the class of strictly increasing, continuous mappings of $[0,1]$ onto itself such that for every $\lambda \in \Lambda$, we have $\lambda(0) = 0$ and $\lambda(1) = 1$. If $X, Y \in D[0,1]$, then the Skorohod metric $\rho(X,Y)$ defining the "distance" between X and Y in $D[0,1]$ is the infimum of those positive ξ for which there exists $\lambda \in \Lambda$ such that*

$$\sup_{t\in[0,1]} |\lambda(t) - t| \leq \xi$$

and

$$\sup_{t\in[0,1]} |X(t) - Y[\lambda(t)]| \leq \xi.$$

See Billingsley [9, p. 111].

Definition A.1.3 *Let* $\mathfrak{D}_\Psi$ *denote the set of discontinuities of the folding operation (A.1.1) in* $D[0,1]$ *so that*

$$\mathfrak{D}_\Psi \equiv \Big\{ x \in D[0,1] : \text{ for some sequence } \{x_n\} \subset D[0,1] \text{ converging to } x, \\ \text{the sequence } \{\Psi_{x_n}\} \text{ does not converge to } \Psi_x \Big\}. \tag{A.1.2}$$

Proposition A.1.4 *If* $\Psi(\cdot) : D[0,1] \to D[0,1]$ *is defined by (A.1.1) with the event* $\mathfrak{D}_\Psi$ *as in (A.1.2), then*

$$\Pr\big\{\mathcal{W}(\cdot) \in \mathfrak{D}_\Psi\big\} = 0 \tag{A.1.3}$$

so that with respect to Wiener measure on $D[0,1]$*, the folding operation* $\Psi(\cdot)$ *is continuous almost surely.*

Proof. We will show that

$$\Pr\big\{\mathcal{W}(\cdot) \in D[0,1] - \mathfrak{D}_\Psi\big\} = 1, \tag{A.1.4}$$

from which (A.1.3) follows immediately. To prove (A.1.4), we need to exploit the almost-sure continuity of sample paths of $\mathcal{W}(\cdot)$:

$$\text{With probability 1, the function } \mathcal{W}(t) \text{ is continuous at every } t \geq 0; \tag{A.1.5}$$

see §41.3.A of Loève [33] or Billingsley [9, p. 64]. Thus we may assume without loss of generality that we are restricting our attention to an event $\mathcal{H} \subset D[0,1]$ for which (A.1.5) holds so that

$$\Pr\big\{\mathcal{W} \in \mathcal{H}\big\} = 1; \tag{A.1.6}$$

and to establish the desired result (A.1.4), we seek to prove that the function $\Psi(\cdot)$ defined by (A.1.1) is continuous at every $\mathcal{W} \in \mathcal{H}$.

Choosing $\mathcal{W} \in \mathcal{H}$ and $\varepsilon > 0$ arbitrarily, we will find $\zeta > 0$ such that

$$\text{For every } Y \in D[0,1] \text{ with } \rho(Y, \mathcal{W}) < \zeta, \text{ we have } \rho\big(\Psi_Y, \Psi_{\mathcal{W}}\big) < \varepsilon; \tag{A.1.7}$$

and then (A.1.6) and (A.1.7) will yield (A.1.4). Throughout the rest of this discussion, we will assume that $\mathcal{W}(\cdot)$ is a fixed sample path in $\mathcal{H}$; and thus virtually all quantities introduced in the rest of this proof depend on the given realization $\mathcal{W}(\cdot)$ of standard Brownian motion.

The sample-path continuity property (A.1.5) and Theorem 4.47 of Apostol [6] imply that $\mathcal{W}(t)$ is uniformly continuous on $[0,1]$; and thus we can find $\zeta_1 > 0$ such that

$$\text{For all } t, t' \in [0,1] \text{ with } |t - t'| < \zeta_1, \text{ we have } |\mathcal{W}(t) - \mathcal{W}(t')| < \varepsilon/4. \tag{A.1.8}$$

Now pick any $Y \in D[0,1]$ such that $\rho(Y, \mathcal{W}) < \min\{\zeta_1, \varepsilon/4\}$. From Definition A.1.2, we see that there is a $\lambda(\cdot) \in \Lambda$ such that

$$\sup_{t \in [0,1]} |\lambda(t) - t| < \min\{\zeta_1, \varepsilon/4\} \tag{A.1.9}$$

and

$$\sup_{t \in [0,1]} |Y(t) - \mathcal{W}[\lambda(t)]| < \min\{\zeta_1, \varepsilon/4\}. \tag{A.1.10}$$

Now for any $t \in [0,1]$, we have

$$\begin{aligned}
&\big|\Psi_Y(t) - \Psi_{\mathcal{W}}(t)\big| \\
&= \big|[Y(t/2) - Y(1-t/2)] - [\mathcal{W}(t/2) - \mathcal{W}(1-t/2)]\big| \\
&\le \big|Y(t/2) - \mathcal{W}(t/2)\big| + \big|Y(1-t/2) - \mathcal{W}(1-t/2)\big| && \text{(A.1.11)} \\
&\le \big|Y(t/2) - \mathcal{W}[\lambda(t/2)]\big| + \big|\mathcal{W}[\lambda(t/2)] - \mathcal{W}(t/2)\big| \\
&\quad + \big|Y(1-t/2) - \mathcal{W}[\lambda(1-t/2)]\big| + \big|\mathcal{W}[\lambda(1-t/2)] - \mathcal{W}(1-t/2)\big| && \text{(A.1.12)} \\
&< \varepsilon/4 + \varepsilon/4 + \varepsilon/4 + \varepsilon/4 = \varepsilon \quad \text{for all } t \in [0,1], && \text{(A.1.13)}
\end{aligned}$$

where (A.1.11) and (A.1.12) follow from two applications of the triangle inequality and (A.1.13) follows from (A.1.9), (A.1.10) and (A.1.8). The required continuity condition (A.1.7) follows from (A.1.13) and Definition A.1.2. $\square$

Applying the FCLT and the Process Limit Lemma in Section 1.2 of Schruben [42], we see that

$$\big[\sqrt{n}\big(\overline{X}_n - \mu_X\big), T_n(\cdot)\big] \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} \big[\sigma Z, \mathcal{B}(\cdot)\big] \tag{A.1.14}$$

in the space $\mathbb{R} \times D[0,1]$, where Z is a standard normal random variable, $\mathcal{B}(\cdot)$ is a Brownian bridge process, and Z and $\mathcal{B}(\cdot)$ are stochastically independent.

Proposition A.1.5 *If Assumptions A hold, then the level-i folded STS converges weakly to a Brownian bridge process,*

$$T_n^{(i)}(\cdot) \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \mathcal{B}(\cdot) \quad \text{for } i = 1, 2, \ldots. \tag{A.1.15}$$

Moreover, for $i = 1, 2, \ldots,$ we have

$$\sqrt{n}\big(\overline{X}_n - \mu_X\big) \text{ and } T_n^{(i)}(\cdot) \text{ are independent as } n \to \infty. \tag{A.1.16}$$

Proof. For each $i \geq 1$, we have

$$T_n^{(i)}(t) = \Psi_{T_n^{(i-1)}}(t) \quad \text{for every } t \in [0,1] \tag{A.1.17}$$

by (A.1.1) and Definitions 2.3.1 and 2.3.3. The proof proceeds by induction on i. Taking $i = 1$ in (A.1.17), we apply Schruben's Process Limit Lemma (A.1.14) and the almost-sure continuity of the folding operation (A.1.3) together with the usual form of the Continuous Mapping Theorem (CMT)—namely, Corollary 1 of Theorem 5.1 of Billingsley [9]—to conclude that

$$T_n^{(1)}(\cdot) = \Psi_{T_n}(\cdot) \underset{n\to\infty}{\overset{\mathcal{D}}{\longrightarrow}} \Psi_{\mathcal{B}}(\cdot) = \mathcal{B}_1(\cdot) \sim \mathcal{B}(\cdot)$$

by Remark 2.1.3. Thus (A.1.15) holds for $i = 1$. Moreover since $T_n^{(1)}(\cdot)$ is a function of $T_n(\cdot)$ alone and $T_n(\cdot)$ is asymptotically independent of $\sqrt{n}\big(\overline{X}_n - \mu_X\big)$, we see that (A.1.16) also holds for $i = 1$. Assuming now that (A.1.15) and (A.1.16) hold for a given $i = i^* > 1$, we can apply (A.1.15)–(A.1.17), (A.1.3), and the CMT to conclude that (A.1.15) and (A.1.16) also hold for $i = i^* + 1$. $\square$

A.2 Proof of Theorem 3.2.1

Definition A.2.1 *For a given weighting function $w(\cdot)$ satisfying Assumptions A and for $n = 1, 2, \ldots,$ the mapping $\Theta_n : Y \in D[0,1] \to \Theta_n(Y) \in \mathbb{R}$ is defined by*

$$\Theta_n(Y) = \frac{1}{n}\sum_{k=1}^{n} w\left(\frac{k}{n}\right) Y\left(\frac{k}{n}\right). \tag{A.2.1}$$

Moreover, the function $\Theta : Y \in D[0,1] \to \Theta(Y) \in \mathbb{R}$ is defined by

$$\Theta(Y) = \int_0^1 w(t)Y(t)\,dt. \tag{A.2.2}$$

Remark A.2.2 The following considerations reveal that the mapping $\Theta(\cdot)$ is well defined. If $Y(\cdot) \in D[0,1]$, then $Y(\cdot)$ has at most countably many discontinuities and is bounded on $[0,1]$; see Billingsley [9, p. 110]. In view of Section 10.11(f) of Rudin [38], we see that $Y(\cdot)$ is continuous almost everywhere with respect to Lebesgue measure; and thus $Y(\cdot)$ is Riemann integrable on $[0,1]$ by Theorem 10.33(b) of Rudin [38]. $\triangleleft$

Next we establish a convergence property of the mappings $\{\Theta_n(\cdot) : n = 1, 2, \ldots\}$ and $\Theta(\cdot)$ that is the key to the asymptotic distribution of the folded weighted area estimator because it sets the stage for applying the extended version of the Continuous Mapping Theorem—namely, Theorem 5.5 of Billingsley [9].

Definition A.2.3 *Let*

$$\mathfrak{D}_\Theta \equiv \Big\{x \in D[0,1] : \text{ for some sequence } \{x_n\} \subset D[0,1] \text{ converging to } x, \\ \text{the sequence } \{\Theta_n(x_n)\} \text{ does not converge to } \Theta(x)\Big\} \tag{A.2.3}$$

denote the set of elements of $D[0,1]$ in which the required convergence property of the mappings $\{\Theta_n(\cdot) : n = 1, 2, \ldots\}$ and $\Theta(\cdot)$ fails to occur.

We establish the following analogue of Proposition A.1.4.

Proposition A.2.4 *If $\Theta_n(\cdot), \Theta(\cdot)$ are defined by (A.2.1) and (A.2.2), respectively, with the event $\mathfrak{D}_\Theta$ as in (A.2.3), then*

$$\Pr\big\{\mathcal{W} \in \mathfrak{D}_\Theta\big\} = 0.$$

Proof. We will show that

$$\Pr\big\{\mathcal{W}(\cdot) \in D[0,1] - \mathfrak{D}_\Theta\big\} = 1. \tag{A.2.4}$$

Proceeding as in the proof of Proposition A.1.4, we assume without loss of generality that we are restricting our attention to an event $\mathcal{H} \subset D[0,1]$ for which the continuity condition (A.1.5) holds for sample paths of Brownian motion so that (A.1.6) also holds; and to establish the desired result (A.2.4), we seek to prove that for every $\mathcal{W} \in \mathcal{H}$ and every sequence $\{x_n\}$ converging to $\mathcal{W}$ in $D[0,1]$, the corresponding sequence $\{\Theta_n(x_n)\}$ converges to $\Theta(\mathcal{W})$ in $\mathbb{R}$.

Choose $\mathcal{W} \in \mathcal{H}$ and $\varepsilon > 0$ arbitrarily. If $\{x_n\} \in D[0,1]$ converges to $\mathcal{W}$, then we will prove that there exists N sufficiently large so that

$$|\Theta_n(x_n) - \Theta(\mathcal{W})| < \varepsilon \quad \text{for every } n \geq N. \tag{A.2.5}$$

By the triangle inequality, we have

$$|\Theta_n(x_n) - \Theta(\mathcal{W})| \leq |\Theta_n(x_n) - \Theta_n(\mathcal{W})| + |\Theta_n(\mathcal{W}) - \Theta(\mathcal{W})| \quad \text{for } n = 1, 2, \ldots . \tag{A.2.6}$$

Our line of attack is to show (i) that the second term on the right-hand side of (A.2.6) tends to zero as $n \to \infty$ because of the Riemann integrability of $\mathcal{W}(\cdot)$ on $[0,1]$; and (ii) that the first term on the right-hand side tends to zero as $n \to \infty$ because $\{x_n\}$ converges to $\mathcal{W}$ and because of the uniform continuity of $\mathcal{W}(\cdot)$ on $[0,1]$.

First we consider the term $|\Theta_n(\mathcal{W}) - \Theta(\mathcal{W})|$ in (A.2.6). Now by the same considerations elaborated in Remark A.2.2 or by the almost-sure continuity of sample paths of Brownian motion, we see that the given realization $\mathcal{W}(\cdot)$ of Brownian motion is Riemann integrable on $[0,1]$; and thus we can find N_1 sufficiently large so that

$$|\Theta_n(\mathcal{W}) - \Theta(\mathcal{W})| = \left| \frac{1}{n} \sum_{k=1}^{n} w\left(\frac{k}{n}\right) \mathcal{W}\left(\frac{k}{n}\right) - \int_0^1 w(t)\mathcal{W}(t)\, dt \right| < \varepsilon/2 \quad \text{for } n \geq N_1. \tag{A.2.7}$$

Next we consider the term $|\Theta_n(x_n) - \Theta_n(\mathcal{W})|$ in (A.2.6). Defining the L_1 norm of the given weighting function,

$$\|w\|_1 = \int_0^1 |w(t)|\, dt, \qquad \text{we have } 0 < \|w\|_1 < \infty$$

by Assumption A.5. It follows that we can find N_2 such that

$$\left| \|w\|_1 - \frac{1}{n} \sum_{k=1}^{n} \left| w\left(\frac{k}{n}\right) \right| \right| \leq \|w\|_1/2 \quad \text{for every } n \geq N_2. \tag{A.2.8}$$

By the uniform continuity of $\mathcal{W}(\cdot)$ on $[0,1]$, we see that we can find $\zeta_2 > 0$ such that

$$\text{For all } t, t' \in [0,1] \text{ with } |t - t'| < \zeta_2, \text{ we have } |\mathcal{W}(t) - \mathcal{W}(t')| < \varepsilon / \big(6\|w\|_1\big). \tag{A.2.9}$$

We have

$$
\begin{aligned}
|\Theta_n(x_n) - \Theta_n(\mathcal{W})| &= \left| \frac{1}{n} \sum_{k=1}^{n} w\left(\frac{k}{n}\right) x_n\left(\frac{k}{n}\right) - \frac{1}{n} \sum_{k=1}^{n} w\left(\frac{k}{n}\right) \mathcal{W}\left(\frac{k}{n}\right) \right| \\
&\le \frac{1}{n} \sum_{k=1}^{n} \left| w\left(\frac{k}{n}\right) \right| \left| x_n\left(\frac{k}{n}\right) - \mathcal{W}\left(\frac{k}{n}\right) \right| \quad \text{for } n = 1, 2, \ldots. \qquad \text{(A.2.10)}
\end{aligned}
$$

Since x_n converges to $\mathcal{W}$ in the Skorohod metric on $D[0,1]$, we can find N_3 such that

$$
\rho(x_n, \mathcal{W}) < \min\left\{\zeta_2, \varepsilon/\left(6\|w\|_1\right)\right\} \quad \text{for every } n \ge N_3. \qquad \text{(A.2.11)}
$$

Combining Definition A.1.2 and (A.2.11), for every $n \ge N_3$ we can find a $\lambda_n(\cdot) \in \Lambda$ such that

$$
\sup_{t\in[0,1]} |\lambda_n(t) - t| < \min\left\{\zeta_2, \varepsilon/\left(6\|w\|_1\right)\right\} \qquad \text{(A.2.12)}
$$

and

$$
\sup_{t\in[0,1]} |x_n(t) - \mathcal{W}[\lambda_n(t)]| < \min\left\{\zeta_2, \varepsilon/\left(6\|w\|_1\right)\right\}. \qquad \text{(A.2.13)}
$$

Now the triangle inequality yields

$$
\begin{aligned}
\left| x_n\left(\frac{k}{n}\right) - \mathcal{W}\left(\frac{k}{n}\right) \right| &\le \left| x_n\left(\frac{k}{n}\right) - \mathcal{W}\left[\lambda_n\left(\frac{k}{n}\right)\right] \right| + \left| \mathcal{W}\left[\lambda_n\left(\frac{k}{n}\right)\right] - \mathcal{W}\left(\frac{k}{n}\right) \right| \\
&< \varepsilon/\left(6\|w\|_1\right) + \varepsilon/\left(6\|w\|_1\right) \qquad \text{(A.2.14)} \\
&= \varepsilon/\left(3\|w\|_1\right) \quad \text{for } n \ge N_3 \text{ and for } k = 1, \ldots, n, \qquad \text{(A.2.15)}
\end{aligned}
$$

where the first term on the right-hand side of (A.2.14) follows from (A.2.13) and the second term on the right-hand side of (A.2.14) follows from (A.2.12) and (A.2.9). Inserting (A.2.15) into (A.2.10), we finally obtain that for $n \ge \max\{N_2, N_3\}$,

$$
\begin{aligned}
|\Theta_n(x_n) - \Theta_n(\mathcal{W})| &< \left[\varepsilon/\left(3\|w\|_1\right)\right] \left[\frac{1}{n} \sum_{k=1}^{n} \left| w\left(\frac{k}{n}\right) \right| \right] \\
&\le \left[\varepsilon/\left(3\|w\|_1\right)\right] \left[\|w\|_1 + \|w\|_1/2\right] \quad \text{by (A.2.8) for } n \ge N_2 \\
&= \varepsilon/2 \quad \text{for } n \ge \max\{N_2, N_3\}; \qquad \text{(A.2.16)}
\end{aligned}
$$

and combining inequalities (A.2.16), (A.2.7), and (A.2.6), we see that (A.2.5) holds with $N = \max\{N_1, N_2, N_3\}$. $\square$

We are finally in position to put all the pieces together.

We see that

$$\sqrt{A_i(w,n)} = N_i(w,n) = \sigma\Theta_n(T_n^{(i)}) \quad \text{for } n = 1, 2, \ldots. \tag{A.2.17}$$

In view of Propositions A.1.5 and A.2.4, we can apply the extended version of the Continuous Mapping Theorem—that is, Theorem 5.5 of Billingsley [9]—and Lemma 2.2.7 to conclude that

$$N_i(w,n) = \sigma\Theta_n\left[T_n^{(i)}\right] \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} \sigma\Theta(\mathcal{B}) = \sigma\int_0^1 w(t)\mathcal{B}(t)\,dt = N_i(w) \sim \sigma\text{Nor}(0,1)$$

so that (3.2.1) follows immediately. Finally the asymptotic independence of $\sqrt{n}\big(\overline{X}_n - \mu_X\big)$ and $A_i(w,n)$ follows immediately from (A.1.16) and (A.2.17), which shows that $A_i(w,n)$ is a function of $T_n^{(i)}(\cdot)$ alone and hence is asymptotically independent of $\sqrt{n}\big(\,\overline{X}_n - \mu_X\,\big)$. $\square$

A.3 Proof of Theorem 4.1.3

Definition A.3.1 *For a given weighting function $g(\cdot)$ satisfying Assumptions A and for $n = 1, 2, \ldots,$ the mapping $\Xi_n : Y \in D[0,1] \to \Xi_n(Y) \in \mathbb{R}$ is defined by*

$$\Xi_n(Y) = \frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right) Y^2\left(\frac{k}{n}\right). \tag{A.3.1}$$

Moreover, the function $\Xi : Y \in D[0,1] \to \Xi(Y) \in \mathbb{R}$ is defined by

$$\Xi(Y) = \int_0^1 g(t)Y^2(t)\,dt. \tag{A.3.2}$$

Remark A.3.2 The following considerations reveal that the mapping $\Xi(\cdot)$ is well defined. If $Y(\cdot) \in D[0,1]$, then $Y^2(\cdot) \in D[0,1]$ has at most countably many discontinuities and is bounded on $[0,1]$; see Billingsley [9, p. 110]. In view of Section 10.11(f) of Rudin [38], we see that $Y^2(\cdot)$ is continuous almost everywhere with respect to Lebesgue measure; and thus $Y^2(\cdot)$ is Riemann integrable on $[0,1]$ by Theorem 10.33(b) of Rudin [38]. $\triangleleft$

Next we establish a convergence property of the mappings $\{\Xi_n(\cdot) : n = 1, 2, \ldots\}$ and $\Xi(\cdot)$ that is the key to the asymptotic distribution of the folded weighted CvM estimator because it sets the stage for applying the extended version of the Continuous Mapping Theorem—namely, Theorem 5.5 of Billingsley [9].

Definition A.3.3 *Let*

$$\boldsymbol{\mathfrak{D}}_{\Xi} \equiv \Big\{ x \in D[0,1] : \text{ for some sequence } \{x_n\} \subset D[0,1] \text{ converging to } x, \\ \text{the sequence } \{\Xi_n(x_n)\} \text{ does not converge to } \Xi(x) \Big\} \tag{A.3.3}$$

denote the set of elements of $D[0,1]$ *in which the required convergence property of the mappings* $\{\Xi_n(\cdot) : n = 1, 2, \ldots\}$ *and* $\Xi(\cdot)$ *fails to occur.*

We establish the following analogue of Proposition A.1.4.

Proposition A.3.4 *If* $\Xi_n(\cdot), \Xi(\cdot)$ *are defined by (A.3.1) and (A.3.2), respectively, with the event* $\boldsymbol{\mathfrak{D}}_{\Xi}$ *as in (A.3.3), then*

$$\Pr\big\{\mathcal{W} \in \boldsymbol{\mathfrak{D}}_{\Xi}\big\} = 0.$$

Proof. We will show that

$$\Pr\big\{\mathcal{W}(\cdot) \in D[0,1] - \boldsymbol{\mathfrak{D}}_{\Xi}\big\} = 1. \tag{A.3.4}$$

Proceeding as in the proof of Proposition A.1.4, we assume without loss of generality that we are restricting our attention to an event $\mathcal{H} \subset D[0,1]$ for which the continuity condition (A.1.5) holds for sample paths of Brownian motion so that (A.1.6) also holds; and to establish the desired result (A.3.4), we seek to prove that for every $\mathcal{W} \in \mathcal{H}$ and every sequence $\{x_n\}$ converging to $\mathcal{W}$ in $D[0,1]$, the corresponding sequence $\{\Xi_n(x_n)\}$ converges to $\Xi(\mathcal{W})$ in $\mathbb{R}$.

Choose $\mathcal{W} \in \mathcal{H}$ and $\varepsilon > 0$ arbitrarily. If $\{x_n\} \in D[0,1]$ converges to $\mathcal{W}$, then we will prove that there exists N sufficiently large so that

$$|\Xi_n(x_n) - \Xi(\mathcal{W})| < \varepsilon \quad \text{ for every } n \geq N. \tag{A.3.5}$$

By the triangle inequality, we have

$$|\Xi_n(x_n) - \Xi(\mathcal{W})| \leq |\Xi_n(x_n) - \Xi_n(\mathcal{W})| + |\Xi_n(\mathcal{W}) - \Xi(\mathcal{W})| \quad \text{ for } n = 1, 2, \ldots . \tag{A.3.6}$$

Our line of attack is to show (i) that the second term on the right-hand side of (A.3.6) tends to zero as $n \to \infty$ because of the Riemann integrability of $\mathcal{W}(\cdot)$ on $[0,1]$; and (ii) that the first term on the right-hand side tends to zero as $n \to \infty$ because $\{x_n\}$ converges to $\mathcal{W}$ and because of the uniform continuity of $\mathcal{W}(\cdot)$ on $[0,1]$.

First we consider the term $|\Xi_n(\mathcal{W}) - \Xi(\mathcal{W})|$ in (A.3.6). Now by the same considerations elaborated in Remark A.3.2 or by the almost-sure continuity of sample paths of Brownian motion, we see that the given realization $\mathcal{W}(\cdot)$ of Brownian motion is Riemann integrable on $[0,1]$; and thus we can find N_1 sufficiently large so that

$$|\Xi_n(\mathcal{W}) - \Xi(\mathcal{W})| = \left| \frac{1}{n} \sum_{k=1}^{n} g\left(\frac{k}{n}\right) \mathcal{W}^2\left(\frac{k}{n}\right) - \int_0^1 g(t)\mathcal{W}^2(t)\, dt \right| < \varepsilon/2 \quad \text{for } n \geq N_1. \tag{A.3.7}$$

Next we consider the term $|\Xi_n(x_n) - \Xi_n(\mathcal{W})|$ in (A.3.6). Defining the L_1 norm of the given weighting function,

$$\|g\|_1 = \int_0^1 |g(t)|\, dt, \qquad \text{we have } 0 < \|g\|_1 < \infty$$

by Assumption A.6. It follows that we can find N_2 such that

$$\left| \|g\|_1 - \frac{1}{n} \sum_{k=1}^{n} \left| g\left(\frac{k}{n}\right) \right| \right| \leq \|g\|_1/2 \quad \text{for every } n \geq N_2. \tag{A.3.8}$$

By the uniform continuity of $\mathcal{W}(\cdot)$ on $[0,1]$, we see that we can find $\zeta_2 > 0$ such that

$$\text{For all } t, t' \in [0,1] \text{ with } |t - t'| < \zeta_2, \text{ we have } |\mathcal{W}(t) - \mathcal{W}(t')| < \varepsilon \big/ \big(6\|g\|_1\big). \tag{A.3.9}$$

We have

$$\begin{aligned} |\Xi_n(x_n) - \Xi_n(\mathcal{W})| &= \left| \frac{1}{n} \sum_{k=1}^{n} g\left(\frac{k}{n}\right) x_n^2\left(\frac{k}{n}\right) - \frac{1}{n} \sum_{k=1}^{n} g\left(\frac{k}{n}\right) \mathcal{W}^2\left(\frac{k}{n}\right) \right| \\ &\leq \frac{1}{n} \sum_{k=1}^{n} \left| g\left(\frac{k}{n}\right) \right| \left| x_n^2\left(\frac{k}{n}\right) - \mathcal{W}^2\left(\frac{k}{n}\right) \right| \quad \text{for } n = 1, 2, \ldots. \end{aligned} \tag{A.3.10}$$

Since x_n converges to $\mathcal{W}$ in the Skorohod metric on $D[0,1]$, x_n^2 converges to $\mathcal{W}^2$ in the same metric, so we can find N_3 such that

$$\rho(x_n^2, \mathcal{W}^2) < \min\big\{\zeta_2, \varepsilon \big/ \big(6\|g\|_1\big)\big\} \quad \text{for every } n \geq N_3. \tag{A.3.11}$$

Combining Definition A.1.2 and (A.3.11), for every $n \geq N_3$ we can find a $\lambda_n(\cdot) \in \Lambda$ such that

$$\sup_{t \in [0,1]} |\lambda_n(t) - t| < \min\big\{\zeta_2, \varepsilon \big/ \big(6\|g\|_1\big)\big\} \tag{A.3.12}$$

and

$$\sup_{t\in[0,1]} |x_n^2(t) - \mathcal{W}^2[\lambda_n(t)]| < \min\bigl\{\zeta_2, \varepsilon/\bigl(6\|g\|_1\bigr)\bigr\}\,. \tag{A.3.13}$$

Now the triangle inequality yields

$$\begin{aligned}
\left|x_n^2\left(\frac{k}{n}\right) - \mathcal{W}^2\left(\frac{k}{n}\right)\right| &\le \left|x_n^2\left(\frac{k}{n}\right) - \mathcal{W}^2\left[\lambda_n\left(\frac{k}{n}\right)\right]\right| + \left|\mathcal{W}^2\left[\lambda_n\left(\frac{k}{n}\right)\right] - \mathcal{W}^2\left(\frac{k}{n}\right)\right| \\
&< \varepsilon/\bigl(6\|g\|_1\bigr) + \varepsilon/\bigl(6\|w\|_1\bigr) \qquad (\text{A.3.14}) \\
&= \varepsilon/\bigl(3\|g\|_1\bigr) \quad \text{for } n \ge N_3 \text{ and for } k = 1, \ldots, n, \qquad (\text{A.3.15})
\end{aligned}$$

where the first term on the right-hand side of (A.3.14) follows from (A.3.13) and the second term on the right-hand side of (A.3.14) follows from (A.3.12) and (A.3.9). Inserting (A.3.15) into (A.3.10), we finally obtain that for $n \ge \max\{N_2, N_3\}$,

$$\begin{aligned}
|\Xi_n(x_n) - \Xi_n(\mathcal{W})| &< \bigl[\varepsilon/\bigl(3\|g\|_1\bigr)\bigr]\left[\frac{1}{n}\sum_{k=1}^{n}\left|g\left(\frac{k}{n}\right)\right|\right] \\
&\le \bigl[\varepsilon/\bigl(3\|g\|_1\bigr)\bigr]\bigl[\|g\|_1 + \|g\|_1/2\bigr] \quad \text{by (A.3.8) for } n \ge N_2 \\
&= \varepsilon/2 \quad \text{ for } n \ge \max\{N_2, N_3\}; \qquad (\text{A.3.16})
\end{aligned}$$

and combining inequalities (A.3.16), (A.3.7), and (A.3.6), we see that (A.3.5) holds with $N = \max\{N_1, N_2, N_3\}$. $\square$

We are finally in position to put all the pieces together.

We see that

$$C_i(g, n) = \sigma\Xi_n(T_n^{(i)}) \quad \text{ for } n = 1, 2, \ldots. \tag{A.3.17}$$

In view of Propositions A.1.5 and A.3.4, we can apply the extended version of the Continuous Mapping Theorem—that is, Theorem 5.5 of Billingsley [9], to conclude that

$$C_i(g, n) = \sigma\Xi_n\left[T_n^{(i)}\right] \underset{n\to\infty}{\xrightarrow{\mathcal{D}}} \sigma\Xi(\mathcal{B}) = \sigma\int_0^1 g(t)\mathcal{B}^2(t)\,dt = C_i(g).$$

so that Theorem 4.1.3 follows immediately. $\square$

APPENDIX B

BIAS AND VARIANCE ASYMPTOTIC RESULTS

This appendix contains the proofs of Theorems 3.2.3 and 4.2.1. Before proving the theorems, we will introduce some notation, and then we state and prove a series of lemmas. First, we define the cumulative sums $Z_k \equiv \sum_{j=1}^{k} X_j$ and the variance time curve $V(k) \equiv \mathrm{Var}(Z_k)$ for $k = 1, 2, \ldots, n$ (see Goldsman and Meketon [25]). Additionally, since the underlying stochastic process satisfies Assumption A.4, we say that R is a positive real constant such that $\sum_{i=1}^{\infty} i^2 \, |R_i| < R$.

B.1 Preliminaries

Lemma B.1.1 *Under Assumptions A.1–A.4,*

$$V(k) = \mathrm{Var}(Z_k) = k\sigma^2 + \gamma - 2\sum_{i=k}^{\infty} (k-i)\, R_i = k\sigma^2 + \gamma + \mathrm{o}(1).$$

Proof: See Goldsman and Meketon [25] (Equation (4)). $\square$

Lemma B.1.2 *Under Assumptions A.1–A.4 and for $k \geq 1$,*

$$\mathrm{E}(Z_k^2) = k\sigma^2 + \gamma + k^2\mu^2 - 2\sum_{i=k}^{\infty}(k-i)R_i.$$

Proof: Apply Lemma B.1.1 to $\mathrm{E}(Z_k^2) = \mathrm{Var}(Z_k) + (\mathrm{E}(Z_k))^2$. $\square$

Lemma B.1.3 *Under Assumptions A.1–A.4 and for $\ell \geq k$,*

$$\mathrm{Cov}(Z_\ell, Z_k) = \frac{1}{2}\left[V(\ell) + V(k) - V(\ell - k)\right].$$

Proof: See Lemma 1 in the Appendix of Goldsman and Meketon [25]. $\square$

Lemma B.1.4 *Under Assumptions A.1–A.4 and for $\ell \geq k$,*

$$\mathrm{Cov}(Z_\ell, Z_k) = k\sigma^2 + \frac{\gamma}{2} - \sum_{i=\ell}^{\infty} (\ell - i)\, R_i - \sum_{i=k}^{\infty} (k-i)\, R_i + \sum_{i=\ell-k}^{\infty} (\ell - k - i)\, R_i.$$

Proof: Combine Lemmas B.1.1 and B.1.3. $\square$

Lemma B.1.5 *Under Assumptions A.1–A.4 and for $\ell \geq k$,*

$$\mathrm{E}(Z_k Z_\ell) = k\sigma^2 + \frac{\gamma}{2} + k\ell\mu^2 - \sum_{i=\ell}^{\infty}(\ell - i)R_i - \sum_{i=k}^{\infty}(k-i)R_i + \sum_{i=\ell-k}^{\infty}(\ell - k - i)R_i.$$

Proof: Apply Lemma B.1.4 to $\mathrm{E}(Z_k Z_\ell) = \mathrm{Cov}(Z_k, Z_\ell) + \mathrm{E}(Z_k)\mathrm{E}(Z_\ell)$. $\square$

B.2 Notation and Auxiliary Lemmas for Theorem 3.2.3

First, we introduce some notation necessary for the proof of Theorem 3.2.3. The subscript D denotes the discrete analogs of previously defined quantities such as $W(t)$ from Theorem 3.2.3. Throughout, we will assume that n is even. Since the weight function $w(\cdot)$ is assumed to be continuous and bounded on $[0,1]$ (see Assumption A.5), we denote $M \equiv \sup_{0\leq t\leq 1}|w(t)| < \infty$, and we define

$$W_{D,n}(t) \equiv \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} w\left(\frac{k}{n}\right), \quad \text{for } 0 \leq t \leq 1. \tag{B.2.1}$$

$$W_{D,n} \equiv W_{D,n}(1) = \frac{1}{n}\sum_{k=1}^{n} w\left(\frac{k}{n}\right). \tag{B.2.2}$$

$$\overline{W}_{D,n} \equiv \frac{1}{n}\sum_{k=1}^{n-1} W_{D,n}\left(\frac{k}{n}\right). \tag{B.2.3}$$

$$\widetilde{w}_n(t) \equiv \begin{cases} w(2t) + w\left(2t + \frac{1}{n}\right) & \text{if } 0 \leq t < \frac{1}{2} - \frac{1}{2n} \\ w(2t) & \text{if } \frac{1}{2} - \frac{1}{2n} < t \leq \frac{1}{2}. \end{cases} \tag{B.2.4}$$

$$\widetilde{W}_{D,n}(t) \equiv \frac{1}{n}\sum_{k=0}^{\lfloor nt\rfloor} \widetilde{w}_n\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{j=0}^{2\lfloor nt\rfloor+1} w\left(\frac{j}{n}\right) \quad \text{for } 0 \leq t \leq 1/2. \tag{B.2.5}$$

$$\breve{w}_n(t) \equiv \begin{cases} w(2t) & \text{if } 0 \leq t \leq \frac{1}{2n} \\ w(2t) + w\left(2t - \frac{1}{n}\right) & \text{if } \frac{1}{2n} < t \leq \frac{1}{2}. \end{cases} \tag{B.2.6}$$

$$\breve{W}_{D,n}(t) \equiv \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor} \breve{w}_n\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{j=1}^{2\lfloor nt\rfloor} w\left(\frac{j}{n}\right) \quad \text{for } 0 \leq t \leq 1/2. \tag{B.2.7}$$

$$R(\ell,k) \equiv -\sum_{i=\ell}^{\infty}(\ell - i)\,R_i - \sum_{i=k}^{\infty}(k-i)\,R_i + \sum_{i=\ell-k}^{\infty}(\ell-k-i)\,R_i \tag{B.2.8}$$

for $\ell = 0, 1, \ldots, n$, $k = 0, 1, \ldots, n$, and $\ell \geq k$.

$$R(\infty, n) \equiv -\sum_{i=n}^{\infty} (n-i)\, R_i. \tag{B.2.9}$$

B.2.1 Auxiliary Lemmas for Theorem 3.2.3

Lemma B.2.1

$$\overline{W}_{D,n} = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{n-j}{n}\right) w\left(\frac{j}{n}\right) = \frac{1}{n^2}\sum_{j=1}^{n}\left(\left\lfloor n-\frac{j}{2}\right\rfloor - \left\lfloor \frac{j}{2}\right\rfloor\right) w\left(\frac{j}{n}\right).$$

Proof: Similar to Foley and Goldsman [20],

$$\overline{W}_{D,n} = \frac{1}{n}\sum_{j=1}^{n-1} W_{D,n}\left(\frac{j}{n}\right) = \frac{1}{n}\sum_{j=1}^{n-1}\frac{1}{n}\sum_{i=1}^{j} w\left(\frac{i}{n}\right) = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{n-j}{n}\right) w\left(\frac{j}{n}\right).$$

The last equality follows directly from the fact that

$$\left\lfloor n-\frac{j}{2}\right\rfloor - \left\lfloor \frac{j}{2}\right\rfloor = n-j \quad \text{for } j = 1, 2, \ldots, n. \quad \square$$

Lemma B.2.2

$$W_{D,n} - \overline{W}_{D,n} = \frac{1}{n^2}\sum_{j=1}^{n} j\, w\left(\frac{j}{n}\right).$$

Proof: By Lemma B.2.1,

$$W_{D,n} - \overline{W}_{D,n} = \frac{1}{n}\sum_{j=1}^{n} w\left(\frac{j}{n}\right) - \frac{1}{n^2}\sum_{j=1}^{n}(n-j) w\left(\frac{j}{n}\right) = \frac{1}{n^2}\sum_{j=1}^{n} j w\left(\frac{j}{n}\right). \quad \square$$

Lemma B.2.3

$$\sum_{j=1}^{n}\left\lfloor\frac{j}{2}\right\rfloor w\left(\frac{j}{n}\right) = \sum_{j=1}^{\frac{n}{2}} j\widetilde{w}_n\left(\frac{j}{n}\right). \tag{B.2.10}$$

Proof: In fact,

$$\sum_{j=1}^{n}\left\lfloor\frac{j}{2}\right\rfloor w\left(\frac{j}{n}\right) = \sum_{j=0}^{\frac{n}{2}} j\left[w\left(\frac{2j}{n}\right) + w\left(\frac{2j+1}{n}\right)\right]$$

(re-indexing and factoring out the new index)

$$= \sum_{j=0}^{\frac{n}{2}} j\widetilde{w}_n\left(\frac{j}{n}\right). \quad \square$$

Lemma B.2.4

$$\frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor n-\frac{j}{2}\right\rfloor \wedge \left\lfloor n-\frac{k}{2}\right\rfloor\right) w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)$$
$$= W_{D,n}^2 - \frac{1}{n^3}\sum_{j=1}^{\frac{n}{2}}\sum_{k=1}^{\frac{n}{2}}(j \vee k)\,\breve{w}_n\left(\frac{j}{n}\right)\breve{w}_n\left(\frac{k}{n}\right).$$

Proof: Since $\left\lfloor n-\frac{k}{2}\right\rfloor = n - \left\lceil \frac{k}{2}\right\rceil$ for $k = 1, 2, \ldots, n$, we have

$$\frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor n-\frac{j}{2}\right\rfloor \wedge \left\lfloor n-\frac{k}{2}\right\rfloor\right) w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)$$
$$= \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left(n-\left\lceil\frac{j}{2}\right\rceil\right) \wedge \left(n-\left\lceil\frac{k}{2}\right\rceil\right)\right) w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)$$
$$= \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(n-\left(\left\lceil\frac{j}{2}\right\rceil \vee \left\lceil\frac{k}{2}\right\rceil\right)\right) w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)$$
$$= \frac{1}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) - \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lceil\frac{j}{2}\right\rceil \vee \left\lceil\frac{k}{2}\right\rceil\right) w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)$$
$$= \left(\frac{1}{n}\sum_{j=1}^{n} w\left(\frac{j}{n}\right)\right)^2 - \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lceil\frac{j}{2}\right\rceil \vee \left\lceil\frac{k}{2}\right\rceil\right) w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)$$
$$= W_{D,n}^2 - \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lceil\frac{j}{2}\right\rceil \vee \left\lceil\frac{k}{2}\right\rceil\right) w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right),$$

where

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lceil\frac{j}{2}\right\rceil \vee \left\lceil\frac{k}{2}\right\rceil\right) w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)$$
$$= \sum_{j=1}^{n}\sum_{k=1}^{n/2}\left(\left\lceil\frac{j}{2}\right\rceil \vee \left\lceil\frac{2k-1}{2}\right\rceil\right) w\left(\frac{j}{n}\right) w\left(\frac{2k-1}{n}\right)$$
$$+ \sum_{j=1}^{n}\sum_{k=1}^{n/2}\left(\left\lceil\frac{j}{2}\right\rceil \vee \left\lceil\frac{2k}{2}\right\rceil\right) w\left(\frac{j}{n}\right) w\left(\frac{2k}{n}\right)$$
$$= \sum_{j=1}^{n/2}\sum_{k=1}^{n/2}\left(\left\lceil\frac{2j-1}{2}\right\rceil \vee k\right) w\left(\frac{2j-1}{n}\right) w\left(\frac{2k-1}{n}\right)$$
$$+ \sum_{j=1}^{n/2}\sum_{k=1}^{n/2}\left(\left\lceil\frac{2j-1}{2}\right\rceil \vee k\right) w\left(\frac{2j-1}{n}\right) w\left(\frac{2k}{n}\right)$$
$$+ \sum_{j=1}^{n/2}\sum_{k=1}^{n/2}\left(\left\lceil\frac{2j}{2}\right\rceil \vee k\right) w\left(\frac{2j}{n}\right) w\left(\frac{2k-1}{n}\right)$$

$$
\begin{aligned}
&+\sum_{j=1}^{n/2}\sum_{k=1}^{n/2}\left(\left\lceil\frac{2j}{2}\right\rceil\vee k\right)w\left(\frac{2j}{n}\right)w\left(\frac{2k}{n}\right)\\
&=\sum_{j=1}^{n/2}\sum_{k=1}^{n/2}(j\vee k)\left[w\left(\frac{2j}{n}\right)+w\left(\frac{2j-1}{n}\right)\right]\left[w\left(\frac{2k}{n}\right)+w\left(\frac{2k-1}{n}\right)\right]\\
&=\sum_{j=1}^{n/2}\sum_{k=1}^{n/2}(j\vee k)\,\breve{w}_n\left(\frac{j}{n}\right)\breve{w}_n\left(\frac{k}{n}\right). \quad \square
\end{aligned}
$$

Lemma B.2.5

$$
\begin{aligned}
&\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor\frac{j}{2}\right\rfloor\vee\left\lfloor\frac{k}{2}\right\rfloor\right)w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)\\
&\quad=\sum_{k=0}^{n/2}\sum_{j=0}^{n/2}(j\vee k)\,\widetilde{w}\left(\frac{j}{n}\right)\widetilde{w}\left(\frac{k}{n}\right)-2w(0)\sum_{j=1}^{n/2}j\widetilde{w}\left(\frac{j}{n}\right).
\end{aligned}
$$

Proof: Observe that

$$
\begin{aligned}
&\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor\frac{j}{2}\right\rfloor\vee\left\lfloor\frac{k}{2}\right\rfloor\right)w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)\\
&=\sum_{j=1}^{n}\sum_{k=0}^{n}\left(\left\lfloor\frac{j}{2}\right\rfloor\vee\left\lfloor\frac{k}{2}\right\rfloor\right)w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)-\sum_{j=1}^{n}\left\lfloor\frac{j}{2}\right\rfloor w\left(\frac{j}{n}\right)w(0)\\
&=\sum_{j=1}^{n}\sum_{k=0}^{n/2}\left(\left\lfloor\frac{j}{2}\right\rfloor\vee\left\lfloor\frac{2k}{2}\right\rfloor\right)w\left(\frac{j}{n}\right)w\left(\frac{2k}{n}\right)\\
&\quad+\sum_{j=1}^{n}\sum_{k=0}^{n/2}\left(\left\lfloor\frac{j}{2}\right\rfloor\vee\left\lfloor\frac{2k+1}{2}\right\rfloor\right)w\left(\frac{j}{n}\right)w\left(\frac{2k+1}{n}\right)-w(0)\sum_{j=1}^{n/2}j\widetilde{w}\left(\frac{j}{n}\right)\\
&\quad\text{(by Lemma B.2.3)}\\
&=\sum_{j=1}^{n}\sum_{k=0}^{n/2}\left(\left\lfloor\frac{j}{2}\right\rfloor\vee k\right)w\left(\frac{j}{n}\right)w\left(\frac{2k}{n}\right)\\
&\quad+\sum_{j=1}^{n}\sum_{k=0}^{n/2}\left(\left\lfloor\frac{j}{2}\right\rfloor\vee k\right)w\left(\frac{j}{n}\right)w\left(\frac{2k+1}{n}\right)-w(0)\sum_{j=1}^{n/2}j\widetilde{w}\left(\frac{j}{n}\right)\\
&=\sum_{j=1}^{n}\sum_{k=0}^{n/2}\left(\left\lfloor\frac{j}{2}\right\rfloor\vee k\right)w\left(\frac{j}{n}\right)\widetilde{w}\left(\frac{k}{n}\right)-w(0)\sum_{j=1}^{n/2}j\widetilde{w}\left(\frac{j}{n}\right)\\
&=\sum_{k=0}^{n/2}\sum_{j=0}^{n/2}\left(\left\lfloor\frac{j}{2}\right\rfloor\vee k\right)w\left(\frac{j}{n}\right)\widetilde{w}\left(\frac{k}{n}\right)-\sum_{k=0}^{n/2}kw(0)\widetilde{w}\left(\frac{k}{n}\right)-w(0)\sum_{j=1}^{n/2}j\widetilde{w}\left(\frac{j}{n}\right)\\
&=\sum_{k=0}^{n/2}\sum_{j=0}^{n/2}\left(\left\lfloor\frac{2j}{2}\right\rfloor\vee k\right)w\left(\frac{2j}{n}\right)\widetilde{w}\left(\frac{k}{n}\right)
\end{aligned}
$$

$$+\sum_{k=0}^{n/2}\sum_{j=0}^{n/2}\left(\left\lfloor\frac{2j+1}{2}\right\rfloor\vee k\right)w\left(\frac{2j+1}{n}\right)\widetilde{w}\left(\frac{k}{n}\right)-2w(0)\sum_{j=1}^{n/2}j\widetilde{w}\left(\frac{j}{n}\right)$$

$$=\sum_{k=0}^{n/2}\sum_{j=0}^{n/2}(j\vee k)\,\widetilde{w}\left(\frac{j}{n}\right)\widetilde{w}\left(\frac{k}{n}\right)-2w(0)\sum_{j=1}^{n/2}j\widetilde{w}\left(\frac{j}{n}\right).\quad\square$$

Lemma B.2.6

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor\frac{j}{2}\right\rfloor\wedge\left\lfloor\frac{k}{2}\right\rfloor\right)w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)$$
$$=2\left(nW_{D,n}+w(0)\right)\sum_{j=1}^{\frac{n}{2}}j\widetilde{w}_n\left(\frac{j}{n}\right)-\sum_{j=0}^{\frac{n}{2}}\sum_{k=0}^{\frac{n}{2}}(j\vee k)\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{w}_n\left(\frac{k}{n}\right).$$

Proof: Observe that

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor\frac{j}{2}\right\rfloor\wedge\left\lfloor\frac{k}{2}\right\rfloor\right)w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{n}\left\{\left(\left\lfloor\frac{j}{2}\right\rfloor\wedge\left\lfloor\frac{k}{2}\right\rfloor\right)+\left(\left\lfloor\frac{j}{2}\right\rfloor\vee\left\lfloor\frac{k}{2}\right\rfloor\right)-\left(\left\lfloor\frac{j}{2}\right\rfloor\vee\left\lfloor\frac{k}{2}\right\rfloor\right)\right\}w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{n}\left\{\left(\left\lfloor\frac{j}{2}\right\rfloor\wedge\left\lfloor\frac{k}{2}\right\rfloor\right)+\left(\left\lfloor\frac{j}{2}\right\rfloor\vee\left\lfloor\frac{k}{2}\right\rfloor\right)\right\}w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)$$
$$-\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor\frac{j}{2}\right\rfloor\vee\left\lfloor\frac{k}{2}\right\rfloor\right)w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)$$
$$=\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor\frac{j}{2}\right\rfloor+\left\lfloor\frac{k}{2}\right\rfloor\right)w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)-\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor\frac{j}{2}\right\rfloor\vee\left\lfloor\frac{k}{2}\right\rfloor\right)w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)$$
$$=2nW_{D,n}\sum_{j=1}^{n}\left\lfloor\frac{j}{2}\right\rfloor w\left(\frac{j}{n}\right)-\sum_{j=0}^{\frac{n}{2}}\sum_{k=0}^{\frac{n}{2}}(j\vee k)\widetilde{w}\left(\frac{j}{n}\right)\widetilde{w}\left(\frac{k}{n}\right)+2w(0)\sum_{j=1}^{\frac{n}{2}}j\widetilde{w}\left(\frac{j}{n}\right),$$

where the last equality is due to Lemma B.2.5. $\square$

Lemma B.2.7 *Under Assumptions A.1–A.4,* $R(\infty,n)=\mathrm{O}(1/n)$.

Proof:

$$\begin{aligned}|R(\infty,n)|&\leq\sum_{i=n}^{\infty}(i-n)\,|R_i|\\&\leq\sum_{i=n}^{\infty}i\,|R_i|\\&=\sum_{i=n}^{\infty}i^2\frac{1}{i}\,|R_i|\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{1}{n} \sum_{i=n}^{\infty} i^2 \, |R_i| \\
&\leq \frac{1}{n} \sum_{i=1}^{\infty} i^2 \, |R_i| \\
&\leq \frac{R}{n},
\end{aligned}
$$

because of Assumption A.4. $\square$

Lemma B.2.8 *Under Assumptions A,*

$$\sum_{j=1}^{n} w\left(\frac{j}{n}\right) R\left(n, \left\lfloor \frac{j}{2} \right\rfloor\right) = \mathrm{O}(1).$$

Proof:

$$
\begin{aligned}
&\frac{1}{M} \left| \sum_{j=1}^{n} w\left(\frac{j}{n}\right) R\left(n, \left\lfloor \frac{j}{2} \right\rfloor\right) \right| \\
&\leq \frac{1}{M} \sum_{j=1}^{n} \left| w\left(\frac{j}{n}\right) \right| \left| R\left(n, \left\lfloor \frac{j}{2} \right\rfloor\right) \right| \\
&\leq \sum_{j=1}^{n} \left[\sum_{i=n}^{\infty} (i-n) \, |R_i| + \sum_{i=\lfloor \frac{j}{2} \rfloor}^{\infty} \left(i - \left\lfloor \frac{j}{2} \right\rfloor \right) |R_i| + \sum_{i=n-\lfloor \frac{j}{2} \rfloor}^{\infty} \left(i - n + \left\lfloor \frac{j}{2} \right\rfloor \right) |R_i| \right] \\
&\qquad \text{(by Assumption A.5 and Equation (B.2.9))} \\
&\leq \sum_{j=1}^{n} \left[\sum_{i=n}^{\infty} i \, |R_i| + \sum_{i=\lfloor \frac{j}{2} \rfloor}^{\infty} i \, |R_i| + \sum_{i=n-\lfloor \frac{j}{2} \rfloor}^{\infty} i \, |R_i| \right] \\
&= n \sum_{i=n}^{\infty} i \, |R_i| + \sum_{j=1}^{n} \sum_{i=\lfloor \frac{j}{2} \rfloor}^{\infty} i \, |R_i| + \sum_{j=1}^{n} \sum_{i=n-\lfloor \frac{j}{2} \rfloor}^{\infty} i \, |R_i| \\
&= n \sum_{i=n}^{\infty} i \, |R_i| \\
&\quad + \left[\sum_{i=\lfloor \frac{1}{2} \rfloor}^{\infty} i \, |R_i| + \sum_{i=\lfloor \frac{2}{2} \rfloor}^{\infty} i \, |R_i| + \sum_{i=\lfloor \frac{3}{2} \rfloor}^{\infty} i \, |R_i| \right. \\
&\qquad \left. + \cdots + \sum_{i=\lfloor \frac{n-2}{2} \rfloor}^{\infty} i \, |R_i| + \sum_{i=\lfloor \frac{n-1}{2} \rfloor}^{\infty} i \, |R_i| + \sum_{i=\lfloor \frac{n}{2} \rfloor}^{\infty} i \, |R_i| \right] \\
&\quad + \left[\sum_{i=n-\lfloor \frac{1}{2} \rfloor}^{\infty} i \, |R_i| + \sum_{i=n-\lfloor \frac{2}{2} \rfloor}^{\infty} i \, |R_i| + \sum_{i=n-\lfloor \frac{3}{2} \rfloor}^{\infty} i \, |R_i| \right.
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \cdots + \sum_{i=n-\left\lfloor \frac{n-2}{2} \right\rfloor}^{\infty} i\,|R_i| + \sum_{i=n-\left\lfloor \frac{n-1}{2} \right\rfloor}^{\infty} i\,|R_i| + \sum_{i=n-\left\lfloor \frac{n}{2} \right\rfloor}^{\infty} i\,|R_i| \Bigg] \\
&= n \sum_{i=n}^{\infty} i\,|R_i| \\
&\quad + \Bigg[\sum_{i=0}^{\infty} i\,|R_i| + \sum_{i=1}^{\infty} i\,|R_i| + \sum_{i=1}^{\infty} i\,|R_i| \\
&\qquad + \cdots + \sum_{i=\frac{n}{2}-1}^{\infty} i\,|R_i| + \sum_{i=\frac{n}{2}-1}^{\infty} i\,|R_i| + \sum_{i=\frac{n}{2}}^{\infty} i\,|R_i| \Bigg] \\
&\quad + \Bigg[\sum_{i=n}^{\infty} i\,|R_i| + \sum_{i=n-1}^{\infty} i\,|R_i| + \sum_{i=n-1}^{\infty} i\,|R_i| \\
&\qquad + \cdots + \sum_{i=\frac{n}{2}+1}^{\infty} i\,|R_i| + \sum_{i=\frac{n}{2}+1}^{\infty} i\,|R_i| + \sum_{i=\frac{n}{2}}^{\infty} i\,|R_i| \Bigg] \\
&= n \sum_{i=n}^{\infty} i\,|R_i| + \sum_{i=0}^{\infty} i\,|R_i| + 2 \sum_{j=1}^{n-1} \sum_{i=j}^{\infty} i\,|R_i| + \sum_{i=n}^{\infty} i\,|R_i| \\
&\leq n \sum_{i=n}^{\infty} i\,|R_i| + 2 \sum_{j=0}^{\infty} \sum_{i=j}^{\infty} i\,|R_i| \\
&\leq \sum_{i=n}^{\infty} i^2\,|R_i| + 2 \sum_{j=0}^{\infty} (j+1) j\,|R_j| \\
&\leq \sum_{i=1}^{\infty} i^2\,|R_i| + 4 \sum_{j=1}^{\infty} j^2\,|R_j| \\
&= 5 \sum_{i=1}^{\infty} i^2\,|R_i| \\
&\leq 5R,
\end{aligned}
$$

by Assumption A.4. □

Lemma B.2.9 *Under Assumptions A,*

$$\sum_{j=1}^{n} w\left(\frac{j}{n}\right) R\left(n, \left\lfloor n - \frac{j}{2} \right\rfloor\right) = \mathrm{O}(1).$$

Proof:

$$
\begin{aligned}
&\frac{1}{M} \left| \sum_{j=1}^{n} w\left(\frac{j}{n}\right) R\left(n, \left\lfloor n - \frac{j}{2} \right\rfloor\right) \right| \\
&\leq \frac{1}{M} \sum_{j=1}^{n} \left| w\left(\frac{j}{n}\right) \right| \left| R\left(n, \left\lfloor n - \frac{j}{2} \right\rfloor\right) \right|
\end{aligned}
$$

$$
\begin{aligned}
&\leq \sum_{j=1}^{n}\left[\sum_{i=n}^{\infty}(i-n)\,|R_i| + \sum_{i=\left\lfloor n-\frac{j}{2}\right\rfloor}^{\infty}\left(i-\left\lfloor n-\frac{j}{2}\right\rfloor\right)|R_i| \right.\\
&\qquad \left. + \sum_{i=n-\left\lfloor n-\frac{j}{2}\right\rfloor}^{\infty}\left(i-n+\left\lfloor n-\frac{j}{2}\right\rfloor\right)|R_i|\right]
\end{aligned}
$$

(by Assumption A.5 and Equation (B.2.9))

$$
\begin{aligned}
&\leq \sum_{j=1}^{n}\left[\sum_{i=n}^{\infty} i\,|R_i| + \sum_{i=\left\lfloor n-\frac{j}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{i=n-\left\lfloor n-\frac{j}{2}\right\rfloor}^{\infty} i\,|R_i|\right]\\
&\leq R + \sum_{j=1}^{n}\sum_{i=\left\lfloor n-\frac{j}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{j=1}^{n}\sum_{i=n-\left\lfloor n-\frac{j}{2}\right\rfloor}^{\infty} i\,|R_i|
\end{aligned}
$$

(by the proof of Lemma B.2.8 and Assumption A.4)

$$
\begin{aligned}
&= R + \sum_{j=1}^{n}\sum_{i=\left\lfloor n-\frac{j}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{j=1}^{n}\sum_{i=n-\left\lfloor n-\frac{j}{2}\right\rfloor}^{\infty} i\,|R_i|\\
&= R + \left[\sum_{i=\left\lfloor n-\frac{1}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{i=\left\lfloor n-\frac{2}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{i=\left\lfloor n-\frac{3}{2}\right\rfloor}^{\infty} i\,|R_i| \right.\\
&\qquad \left. + \cdots + \sum_{i=\left\lfloor n-\frac{n-2}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{i=\left\lfloor n-\frac{n-1}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{i=\left\lfloor n-\frac{n}{2}\right\rfloor}^{\infty} i\,|R_i|\right]\\
&\quad + \left[\sum_{i=n-\left\lfloor n-\frac{1}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{i=n-\left\lfloor n-\frac{2}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{i=n-\left\lfloor n-\frac{3}{2}\right\rfloor}^{\infty} i\,|R_i| \right.\\
&\qquad \left. + \cdots + \sum_{i=n-\left\lfloor n-\frac{n-2}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{i=n-\left\lfloor n-\frac{n-1}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{i=n-\left\lfloor n-\frac{n}{2}\right\rfloor}^{\infty} i\,|R_i|\right]\\
&= R + \left[\sum_{i=n-1}^{\infty} i\,|R_i| + \sum_{i=n-1}^{\infty} i\,|R_i| + \sum_{i=n-2}^{\infty} i\,|R_i| + \sum_{i=n-2}^{\infty} i\,|R_i| \right.\\
&\qquad \left. + \cdots + \sum_{i=\frac{n}{2}-1}^{\infty} i\,|R_i| + \sum_{i=\frac{n}{2}}^{\infty} i\,|R_i| + \sum_{i=\frac{n}{2}}^{\infty} i\,|R_i|\right]\\
&\quad + \left[\sum_{i=1}^{\infty} i\,|R_i| + \sum_{i=1}^{\infty} i\,|R_i| + \sum_{i=2}^{\infty} i\,|R_i| + \sum_{i=2}^{\infty} i\,|R_i| \right.\\
&\qquad \left. + \cdots + \sum_{i=\frac{n}{2}}^{\infty} i\,|R_i| + \sum_{i=\frac{n}{2}}^{\infty} i\,|R_i|\right]\\
&= R + 2\sum_{j=1}^{n-1}\sum_{i=j}^{\infty} i\,|R_i| + 2\sum_{i=\frac{n}{2}}^{\infty} i\,|R_i|
\end{aligned}
$$

$$\leq R + 8R + \frac{4R}{n}$$

$$\leq 13R,$$

by Assumption A.4 and the proof of Lemma B.2.8. $\square$

Lemma B.2.10 *Under Assumptions A,*

$$\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) R\left(\left\lfloor n-\frac{k}{2}\right\rfloor, \left\lfloor \frac{j}{2}\right\rfloor\right) = \mathrm{O}(n).$$

Proof: As in the proof of Lemma B.2.9,

$$\frac{1}{M^2}\left|\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) R\left(\left\lfloor n-\frac{k}{2}\right\rfloor, \left\lfloor \frac{j}{2}\right\rfloor\right)\right|$$

$$\leq \sum_{j=1}^{n}\sum_{k=1}^{n}\left[\sum_{i=\lfloor n-\frac{k}{2}\rfloor}^{\infty} i\,|R_i| + \sum_{i=\lfloor \frac{j}{2}\rfloor}^{\infty} i\,|R_i| + \sum_{i=\lfloor n-\frac{k}{2}\rfloor-\lfloor\frac{j}{2}\rfloor}^{\infty} i\,|R_i|\right]$$

(by Assumption A.5 and Equation (B.2.9))

$$= n\sum_{k=1}^{n}\sum_{i=\lfloor n-\frac{k}{2}\rfloor}^{\infty} i\,|R_i| + n\sum_{j=1}^{n}\sum_{i=\lfloor \frac{j}{2}\rfloor}^{\infty} i\,|R_i| + \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=\lfloor n-\frac{k}{2}\rfloor-\lfloor\frac{j}{2}\rfloor}^{\infty} i\,|R_i|.$$

Since the first two sums were shown to be $\mathrm{O}(n)$ during the proof of Lemmas B.2.9, we will concentrate on the last term alone. In particular,

$$\sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=\lfloor n-\frac{k}{2}\rfloor-\lfloor\frac{j}{2}\rfloor}^{\infty} i\,|R_i| = \sum_{j=1}^{n}\sum_{k=1}^{n}\sum_{i=n-\lceil\frac{k}{2}\rceil-\lfloor\frac{j}{2}\rfloor}^{\infty} i\,|R_i|$$

$$= \sum_{j=1}^{n}\sum_{i=n-\lfloor\frac{j}{2}\rfloor-1}^{\infty} ni\,|R_i| + (n-2)\sum_{j=1}^{n}\left(n-2-\left\lfloor\frac{j}{2}\right\rfloor\right)\left|R_{n-2-\lfloor\frac{j}{2}\rfloor}\right|$$

$$+ (n-4)\sum_{j=1}^{n}\left(n-3-\left\lfloor\frac{j}{2}\right\rfloor\right)\left|R_{n-3-\lfloor\frac{j}{2}\rfloor}\right|$$

$$+ \cdots + 2\sum_{j=1}^{n}\left(n-\frac{n}{2}-\left\lfloor\frac{j}{2}\right\rfloor\right)\left|R_{n-\frac{n}{2}-\lfloor\frac{j}{2}\rfloor}\right|,$$

where the last equality is obtained by displaying the two inner sums in different rows (one row for each value of k) and adding columns instead of such rows. By observing the common pattern in the previous sums, we find that the previous expression equals

$$
\begin{aligned}
&\sum_{j=1}^{n} \sum_{i=n-\left\lfloor \frac{j}{2} \right\rfloor -1}^{\infty} ni\,|R_i| + \sum_{j=1}^{n} \sum_{i=\frac{n}{2}-\left\lfloor \frac{j}{2} \right\rfloor}^{n-\left\lfloor \frac{j}{2} \right\rfloor -2} \left(2i + 2\left\lfloor \frac{j}{2} \right\rfloor - (n-2)\right) i\,|R_i| \\
&\leq \sum_{j=1}^{n} \sum_{i=n-\left\lfloor \frac{j}{2} \right\rfloor -1}^{\infty} ni\,|R_i| + \sum_{j=1}^{n} \sum_{i=\frac{n}{2}-\left\lfloor \frac{j}{2} \right\rfloor}^{n-\left\lfloor \frac{j}{2} \right\rfloor -2} ni\,|R_i| \\
&= \sum_{j=1}^{n} \sum_{i=\frac{n}{2}-\left\lfloor \frac{j}{2} \right\rfloor}^{\infty} ni\,|R_i| \\
&\leq \sum_{j=1}^{n} \sum_{i=\left\lfloor \frac{j}{2} \right\rfloor}^{\infty} ni\,|R_i| \\
&\leq 2 \sum_{j=0}^{\frac{n}{2}} \sum_{i=j}^{\infty} ni\,|R_i| \\
&\leq 4nR,
\end{aligned}
$$

by Assumption A.4 and the proof of Lemma B.2.8. $\square$

Lemma B.2.11 *Under Assumptions A,*

$$
\sum_{j=2}^{n} \sum_{k=1}^{j-1} \sum_{i=\left\lfloor \frac{j}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor}^{\infty} i\,|R_i| = \mathrm{O}(n).
$$

Proof: By exchanging the order of the two outer sums, we have

$$
\sum_{j=2}^{n} \sum_{k=1}^{j-1} \sum_{i=\left\lfloor \frac{j}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor}^{\infty} i\,|R_i| = \sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \sum_{i=\left\lfloor \frac{j}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor}^{\infty} i\,|R_i|, \tag{B.2.11}
$$

where

$$
\begin{aligned}
&\sum_{j=k+1}^{n} \sum_{i=\left\lfloor \frac{j}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor}^{\infty} i\,|R_i| \\
&= \sum_{j=1}^{n} \sum_{i=\left\lfloor \frac{j}{2} \right\rfloor}^{\infty} ni\,|R_i| \\
&= \sum_{j=k+1}^{n-1} \sum_{i=\left\lfloor \frac{j}{2} \right\rfloor - \left\lfloor \frac{k}{2} \right\rfloor}^{\frac{n}{2}-\left\lfloor \frac{k}{2} \right\rfloor -1} i\,|R_i| + \sum_{j=k+1}^{n-1} \sum_{i=\frac{n}{2}-\left\lfloor \frac{k}{2} \right\rfloor}^{\infty} i\,|R_i| + \sum_{i=\frac{n}{2}-\left\lfloor \frac{k}{2} \right\rfloor}^{\infty} i\,|R_i| \\
&= 2 \sum_{i=1}^{\frac{n}{2}-\left\lfloor \frac{k}{2} \right\rfloor -1} i\,|R_i| + 2 \sum_{i=2}^{\frac{n}{2}-\left\lfloor \frac{k}{2} \right\rfloor -1} i\,|R_i| + \cdots + 2\left(\frac{n}{2} - \left\lfloor \frac{k}{2} \right\rfloor - 1\right) \left|R_{\frac{n}{2}-\left\lfloor \frac{k}{2} \right\rfloor -1}\right|
\end{aligned}
$$

$$+ (n-k-2) \sum_{i=\frac{n}{2}-\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{i=\frac{n}{2}-\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} i\,|R_i|$$

$$= 2(1\,|R_1|) + 4(2\,|R_2|)$$

$$+ \cdots + \left(\frac{n}{2} - \left\lfloor\frac{k}{2}\right\rfloor - 1\right)\left(2\left(\frac{n}{2} - \left\lfloor\frac{k}{2}\right\rfloor - 1\right)\left|R_{\frac{n}{2}-\left\lfloor\frac{k}{2}\right\rfloor-1}\right|\right)$$

$$+ (n-k-2) \sum_{i=\frac{n}{2}-\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{i=\frac{n}{2}-\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} i\,|R_i|$$

$$= \sum_{k=1}^{n-1} \sum_{i=1}^{\frac{n}{2}-\left\lfloor\frac{k}{2}\right\rfloor-1} (2i)i\,|R_i| + (n-k-2) \sum_{i=\frac{n}{2}-\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{i=\frac{n}{2}-\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} i\,|R_i|. \tag{B.2.12}$$

Substituting Equation (B.2.12) into Equation (B.2.11)

$$\sum_{j=2}^{n}\sum_{k=1}^{j-1} \sum_{i=\left\lfloor\frac{j}{2}\right\rfloor-\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} i\,|R_i|$$

$$= 2\sum_{k=1}^{n-1} \sum_{i=1}^{\frac{n}{2}-\left\lfloor\frac{k}{2}\right\rfloor-1} i^2\,|R_i| + \sum_{k=1}^{n-1}(n-k-2) \sum_{i=\frac{n}{2}-\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} i\,|R_i| + \sum_{k=1}^{n-1} \sum_{i=\frac{n}{2}-\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} i\,|R_i|$$

$$\leq 2n\sum_{i=1}^{\infty} i^2\,|R_i| + \sum_{k=1}^{n-1}(n-k-2) \sum_{i=\frac{n}{2}-\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} i^2\,|R_i|\,\frac{1}{i} + \sum_{k=1}^{n-1} \sum_{i=\frac{n}{2}-\left\lfloor\frac{k}{2}\right\rfloor}^{\infty} i\,|R_i|$$

$$\leq 2nR + \sum_{k=1}^{n-2}(n-k-2)\left(\frac{1}{\frac{n}{2} - \left\lfloor\frac{k}{2}\right\rfloor}\right)\sum_{i=1}^{\infty} i^2\,|R_i| + 4R$$

(by Assumption A.4 and the proof of Lemma B.2.10)

$$= 2nR + \sum_{k=1}^{n-3}(n-k-2)\left(\frac{1}{\frac{n}{2} - \left\lfloor\frac{k}{2}\right\rfloor}\right)\sum_{i=1}^{\infty} i^2\,|R_i| + 4R.$$

Now, observe that for every integer k such that $1 \leq k \leq n-3$, we have

$$\frac{n-k-2}{\frac{n}{2} - \left\lfloor\frac{k}{2}\right\rfloor} \leq \frac{n-k-2}{\frac{n}{2} - \left\lceil\frac{k}{2}\right\rceil} \leq \frac{n-k-2}{\frac{n}{2} - \frac{k}{2} - 1} = 2. \tag{B.2.13}$$

Then

$$2nR + \sum_{k=1}^{n-3} \frac{n-k-2}{\frac{n}{2} - \left\lfloor\frac{k}{2}\right\rfloor} \sum_{i=1}^{\infty} i^2\,|R_i| + 4R \leq 8nR. \quad \Box$$

Lemma B.2.12 *Under Assumptions A,*

$$\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) R\left(\left\lfloor\frac{j}{2}\right\rfloor \vee \left\lfloor\frac{k}{2}\right\rfloor, \left\lfloor\frac{j}{2}\right\rfloor \wedge \left\lfloor\frac{k}{2}\right\rfloor\right) = \mathrm{O}(n).$$

Proof: As in the proof of Lemma B.2.9,

$$\frac{1}{M^2}\left|\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) R\left(\left\lfloor\frac{j}{2}\right\rfloor \vee \left\lfloor\frac{k}{2}\right\rfloor, \left\lfloor\frac{j}{2}\right\rfloor \wedge \left\lfloor\frac{k}{2}\right\rfloor\right)\right|$$

$$\le \sum_{j=1}^{n}\sum_{k=1}^{n}\left[\sum_{i=\lfloor\frac{j}{2}\rfloor\vee\lfloor\frac{k}{2}\rfloor}^{\infty} i\,|R_i| + \sum_{i=\lfloor\frac{j}{2}\rfloor\wedge\lfloor\frac{k}{2}\rfloor}^{\infty} i\,|R_i| + \sum_{i=(\lfloor\frac{j}{2}\rfloor\vee\lfloor\frac{k}{2}\rfloor)-(\lfloor\frac{j}{2}\rfloor\wedge\lfloor\frac{k}{2}\rfloor)}^{\infty} i\,|R_i|\right]$$

$$= \underbrace{2\sum_{k=1}^{n}\left[\sum_{i=\lfloor\frac{k}{2}\rfloor}^{\infty} i\,|R_i| + \sum_{i=1}^{\infty} i\,|R_i|\right]}_{\text{(case } j=k\text{)}}$$

$$+ \underbrace{2\sum_{j=2}^{n}\sum_{k=1}^{j-1}\left[\sum_{i=\lfloor\frac{j}{2}\rfloor}^{\infty} i\,|R_i| + \sum_{i=\lfloor\frac{k}{2}\rfloor}^{\infty} i\,|R_i| + \sum_{i=\lfloor\frac{j}{2}\rfloor-\lfloor\frac{k}{2}\rfloor}^{\infty} i\,|R_i|\right]}_{\text{(case } j>k \text{ and symmetry)}}.$$

$$= \underbrace{2\sum_{k=1}^{n}\sum_{i=\lfloor\frac{k}{2}\rfloor}^{\infty} i\,|R_i|}_{(\le 4R \text{ by proof of Lemma B.2.8})} + \underbrace{2n\sum_{i=1}^{\infty} i\,|R_i|}_{(\le 2nR \text{ by Assumption A.4})}$$

$$+ \underbrace{2\sum_{j=2}^{n}(j-1)\sum_{i=\lfloor\frac{j}{2}\rfloor}^{\infty} i\,|R_i|}_{(\le 4nR \text{ by Lemma B.2.8})} + 2\sum_{j=2}^{n}\underbrace{\sum_{k=1}^{j-1}\sum_{i=\lfloor\frac{k}{2}\rfloor}^{\infty} i\,|R_i|}_{(\le 2R \text{ by Lemma B.2.8})}$$

$$+ \underbrace{2\sum_{j=2}^{n}\sum_{k=1}^{j-1}\sum_{i=\lfloor\frac{j}{2}\rfloor-\lfloor\frac{k}{2}\rfloor}^{\infty} i\,|R_i|}_{(\le 8nR \text{ by Lemma B.2.11})}$$

$$\le 22nR. \quad \square$$

Lemma B.2.13 *Under Assumptions A,*

$$\sum_{j=2}^{n}\sum_{k=1}^{j-1}\sum_{i=\lfloor n-\frac{j}{2}\rfloor-\lfloor n-\frac{k}{2}\rfloor}^{\infty} i\,|R_i| = \mathrm{O}(n).$$

Proof: Changing the order of the two outer sums, we get

$$\sum_{j=2}^{n}\sum_{k=1}^{j-1}\sum_{i=\lfloor n-\frac{j}{2}\rfloor-\lfloor n-\frac{k}{2}\rfloor}^{\infty} i\,|R_i| = \sum_{k=1}^{n-1}\sum_{j=k+1}^{n}\sum_{i=\lfloor n-\frac{j}{2}\rfloor-\lfloor n-\frac{k}{2}\rfloor}^{\infty} i\,|R_i|$$

$$= \sum_{k=1}^{n-1}\sum_{j=k+1}^{n}\sum_{i=\lceil\frac{j}{2}\rceil-\lceil\frac{k}{2}\rceil}^{\infty} i\,|R_i|,$$

where

$$\sum_{j=k+1}^{n} \sum_{i=\lceil \frac{j}{2} \rceil - \lceil \frac{k}{2} \rceil}^{\infty} i\,|R_i| = 2 \sum_{i=1}^{\frac{n}{2}-\lceil \frac{k}{2} \rceil - 1} i^2\,|R_i| + (n-k-1) \sum_{i=\frac{n}{2}-\lceil \frac{k}{2} \rceil}^{\infty} i\,|R_i|\,.$$

Hence,

$$\begin{aligned}
&\sum_{k=1}^{n-1} \sum_{j=k+1}^{n} \sum_{i=\lceil \frac{j}{2} \rceil - \lceil \frac{k}{2} \rceil}^{\infty} i\,|R_i| \\
&= \sum_{k=1}^{n-1} \left[2 \sum_{i=1}^{\frac{n}{2}-\lceil \frac{k}{2} \rceil - 1} i^2\,|R_i| + (n-k-1) \sum_{i=\frac{n}{2}-\lceil \frac{k}{2} \rceil}^{\infty} i\,|R_i| \right] \\
&\leq 2n \sum_{i=1}^{\infty} i^2\,|R_i| + \sum_{k=1}^{n-1} (n-k-1) \sum_{i=\frac{n}{2}-\lceil \frac{k}{2} \rceil}^{\infty} i^2\,|R_i|\,\frac{1}{i} \\
&\leq 2nR + \sum_{k=1}^{n-2} (n-k-1) \frac{1}{\frac{n}{2} - \lceil \frac{k}{2} \rceil} \sum_{i=\frac{n}{2}-\lceil \frac{k}{2} \rceil}^{\infty} i^2\,|R_i| \\
&\leq 2nR + \sum_{k=1}^{n-2} (n-k-1) \frac{1}{\frac{n}{2} - \frac{k}{2} - \frac{1}{2}} \sum_{i=1}^{\infty} i^2\,|R_i| \\
&\leq 4nR. \quad \square
\end{aligned}$$

Lemma B.2.14 *Under Assumptions A,*

$$\sum_{j=1}^{n} \sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) R\left(\left\lfloor n - \frac{j}{2} \right\rfloor \vee \left\lfloor n - \frac{k}{2} \right\rfloor, \left\lfloor n - \frac{j}{2} \right\rfloor \wedge \left\lfloor n - \frac{k}{2} \right\rfloor\right) = \mathrm{O}(n).$$

Proof: It parallels the proof of Lemma B.2.12 by using Lemma B.2.13 instead of Lemma B.2.11.

$$\begin{aligned}
&\frac{1}{M^2} \left| \sum_{j=1}^{n} \sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) R\left(\left\lfloor n - \frac{j}{2} \right\rfloor \vee \left\lfloor n - \frac{k}{2} \right\rfloor, \left\lfloor n - \frac{j}{2} \right\rfloor \wedge \left\lfloor n - \frac{k}{2} \right\rfloor\right) \right| \\
&\leq \sum_{j=1}^{n} \sum_{k=1}^{n} \left[\sum_{i=\lfloor n-\frac{j}{2} \rfloor \vee \lfloor n-\frac{k}{2} \rfloor}^{\infty} i\,|R_i| \sum_{i=\lfloor n-\frac{j}{2} \rfloor \wedge \lfloor n-\frac{k}{2} \rfloor}^{\infty} i\,|R_i| \right. \\
&\qquad \left. \sum_{i=(\lfloor n-\frac{j}{2} \rfloor \vee \lfloor n-\frac{k}{2} \rfloor)-(\lfloor n-\frac{j}{2} \rfloor \wedge \lfloor n-\frac{k}{2} \rfloor)}^{\infty} i\,|R_i| \right] \\
&= \sum_{k=1}^{n} \left[2 \sum_{i=\lfloor n-\frac{k}{2} \rfloor}^{\infty} i\,|R_i| + \sum_{i=1}^{\infty} i\,|R_i| \right]
\end{aligned}$$

(case $j = k$)

$$+\, 2 \sum_{j=2}^{n} \sum_{k=1}^{j-1} \left[\sum_{i=\lfloor n-\frac{k}{2} \rfloor}^{\infty} i\,|R_i| + \sum_{i=\lfloor n-\frac{j}{2} \rfloor}^{\infty} i\,|R_i| + \sum_{i=\lfloor n-\frac{k}{2} \rfloor - \lfloor n-\frac{j}{2} \rfloor}^{\infty} i\,|R_i| \right]$$

(case $j \geq k$ and symmetry)

$$= 2 \sum_{i=\frac{n}{2}}^{\infty} [n \wedge (2i-n+2)] i\,|R_i| + 2n \sum_{i=1}^{\infty} i\,|R_i|$$

$$+\, 2 \sum_{j=2}^{n} \sum_{i=\lfloor \frac{j}{2} \rfloor}^{\infty} [(j-1) \wedge (2i-j+3)] i\,|R_i|$$

$$+\, 2 \sum_{j=2}^{n} (j-1) \sum_{i=\lfloor n-\frac{j}{2} \rfloor}^{\infty} i\,|R_i| + 2 \sum_{j=2}^{n} \sum_{k=1}^{j-1} \sum_{i=\lfloor n-\frac{k}{2} \rfloor - \lfloor n-\frac{j}{2} \rfloor}^{\infty} i\,|R_i|$$

$$\leq 8 \sum_{i=\frac{n}{2}}^{\infty} i^2\,|R_i| + 2n \sum_{i=1}^{\infty} i^2\,|R_i|$$

$$+\, 10n \sum_{i=1}^{\infty} i^2\,|R_i| + 2 \sum_{i=\frac{n}{2}}^{\infty} \left[\frac{(n-1)n}{2} \wedge \sum_{j=2(n-i-1)}^{n-1} j \right] i\,|R_i|$$

$$+\, 2 \sum_{j=2}^{n} \sum_{k=1}^{j-1} \sum_{i=\lfloor n-\frac{j}{2} \rfloor - \lfloor n-\frac{k}{2} \rfloor}^{\infty} i\,|R_i|.$$

Now, notice that

$$2 \sum_{i=\frac{n}{2}}^{\infty} \left[\frac{(n-1)n}{2} \wedge \sum_{j=2(n-i-1)}^{n-1} j \right] i\,|R_i| \leq n^2 \sum_{i=\frac{n}{2}}^{\infty} i^2\,|R_i| \frac{1}{i} \leq 2n \sum_{i=1}^{\infty} i^2\,|R_i| \leq 2nR.$$

Finally, by invoking Lemma B.2.13, we can conclude that

$$\left| \sum_{j=1}^{n} \sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) R\left(\left\lfloor n - \frac{j}{2} \right\rfloor \vee \left\lfloor n - \frac{k}{2} \right\rfloor, \left\lfloor n - \frac{j}{2} \right\rfloor \wedge \left\lfloor n - \frac{k}{2} \right\rfloor \right) \right| \leq 28M^2 nR. \quad \square$$

During the proof of Theorem 3.2.3, we would like to replace the discrete approximations to integrals with their respective integrals. To do this, we need some "big-Oh" expressions relating the approximations and the integrals. We set up these results with the following lemmas and the Trapezoid Rule for integrals which we state next.

Trapezoid Rule for Integrals (See Atkinson ([7, p. 253])) Let $f(\cdot)$ be a continuous function on $[a, b]$ with at least two continuous derivatives on $[a, b]$. Let

$$I(f) \equiv \int_a^b f(t)\, dt.$$

Then

$$I(f) = \frac{h}{2}\sum_{k=1}^{n}[f(x_{k-1}) + f(x_k)] - \frac{h^3}{12}\sum_{k=1}^{n} f''(\eta_k)$$
$$= \frac{h}{2}[f(a) + f(b)] + h\sum_{k=1}^{n-1} f(x_k) - \frac{h^3}{12}\sum_{k=1}^{n} f''(\eta_k), \tag{B.2.14}$$

where $h \equiv (b-a)/n$, $x_k \equiv a + kh$, and $\eta_k \in [x_{k-1}, x_k]$, for $k = 1, 2, \ldots, n$.

Remark B.2.15 Since $f''(\cdot)$ is continuous and bounded on $[a, b]$,

$$I(f) = \frac{h}{2}[f(a) + f(b)] + h\sum_{k=1}^{n-1} f(x_k) + \mathrm{O}(1/n^2). \quad \triangleleft \tag{B.2.15}$$

Lemma B.2.16 *Under Assumption A.5,*

$$W_{D,n}^2 = W^2 + \frac{(w(1) - w(0))W}{n} + \mathrm{O}(1/n^2) = W^2 + \mathrm{O}(1/n).$$

Proof: By the trapezoid rule for integrals

$$\begin{aligned}
W &= \int_0^1 w(t)\,dt \\
&= \frac{1}{2n}(w(0) + w(1)) + \frac{1}{n}\sum_{k=1}^{n-1} w\left(\frac{k}{n}\right) - \frac{1}{12n^3}\sum_{k=1}^{n} w''(\eta_k) \\
&\qquad \text{(for some } (k-1)/n \le \eta_k \le k/n \text{ where } k = 1, 2, \ldots, n) \\
&= \frac{1}{2n}(w(0) - w(1)) + W_{D,n} + \mathrm{O}(1/n^2) \\
&\qquad \text{(by adding and substracting } w(1)/n) \\
&= W_{D,n} + \mathrm{O}(1/n).
\end{aligned} \tag{B.2.16}$$

Squaring both sides of Equation (B.2.16), we have

$$W_{D,n}^2 = W^2 + \mathrm{O}(1/n). \quad \square$$

Lemma B.2.17 *Under Assumption A.5,*

$$\overline{W}_{D,n}^2 = \overline{W}^2 - \frac{2w(0)\overline{W}}{n} + \mathrm{O}(1/n^2) = \overline{W}^2 + \mathrm{O}(1/n). \tag{B.2.17}$$

Proof: Starting with the definition of $\overline{W}$ and applying the Trapezoid Rule for integrals to it we have

$$\begin{aligned}
\overline{W} &= \int_0^1 W(t)\,dt \\
&= \frac{1}{2n}(W(0)+W(1)) + \frac{1}{n}\sum_{k=1}^{n-1} W\left(\frac{k}{n}\right) - \frac{1}{12n^3}\sum_{k=1}^{n} W''(\eta_k) \\
&\qquad \text{(for some } (k-1)/n \le \eta_k \le k/n \text{ with } k = 1,2,\ldots,n) \\
&= \frac{W}{2n} + \frac{1}{n}\sum_{k=1}^{n-1} W\left(\frac{k}{n}\right) - \frac{1}{12n^3}\sum_{k=1}^{n} w'(\eta_k) \\
&= -\frac{W}{2n} + \frac{1}{n}\sum_{k=1}^{n} W\left(\frac{k}{n}\right) + \mathrm{O}(1/n^2) \\
&\qquad \text{(by adding and substracting } W/n\text{, and Assumption A.5)} \\
&= -\frac{\frac{w(0)-w(1)}{2n} + W_{D,n} + \mathrm{O}(1/n^2)}{2n} + \frac{1}{n}\sum_{k=1}^{n}\left[\frac{w(0)-w(k/n)}{2n} + W_{D,n}\left(\frac{k}{n}\right) + \mathrm{O}(1/n^2)\right] \\
&\qquad \text{(Assumption A.5 and applying the Trapezoid Rule to } W \text{ and } W(k/n)) \\
&= \frac{1}{n}\sum_{k=1}^{n-1} W_{D,n}\left(\frac{k}{n}\right) + \frac{w(0)}{n} + \mathrm{O}(1/n^2) \\
&= \overline{W}_{D,n} + \frac{w(0)}{n} + \mathrm{O}(1/n^2) \\
&= \overline{W}_{D,n} + \mathrm{O}(1/n). \qquad \text{(B.2.18)}
\end{aligned}$$

The proof follows by squaring both sides of Equation (B.2.18). □

In the following lemma we express $\widetilde{W}_{D,n}(1/2)$ in terms of the integral W.

Lemma B.2.18 *Under Assumption A.5,*

$$\widetilde{W}_{D,n}(1/2) = W + \frac{w(0)+w(1)}{2n} + \mathrm{O}(1/n^2).$$

Proof: We have

$$\begin{aligned}
\widetilde{W}_{D,n}(1/2) &= \frac{1}{n}\sum_{k=0}^{n/2} \widetilde{w}_n\left(\frac{k}{n}\right) \\
&= \frac{1}{n}\sum_{k=0}^{n+1} w\left(\frac{k}{n}\right) \\
&= \frac{1}{n}\sum_{k=0}^{n} w\left(\frac{k}{n}\right)
\end{aligned}$$

$$\begin{aligned}
&\quad \text{(since } w(u) = 0 \text{ for } u \geq (1 + 1/n))\\
&= W_{D,n} + \frac{w(0)}{n}\\
&= \frac{w(0) + w(1)}{2n} + W + \mathrm{O}(1/n^2),
\end{aligned}$$

by the Trapezoid Rule for integrals and Assumption A.5. $\square$

Lemma B.2.19 *Under Assumption A.5,*

$$\frac{1}{n} \sum_{k=1}^{\frac{n}{2}} \frac{k}{n} \widetilde{w}_n \left(\frac{k}{n}\right) = \int_0^{1/2} u \widehat{w}_n(u)\, du + \frac{w(1) - w(1 + 1/n)}{4n} + \mathrm{O}(1/n^2).$$

where $\widehat{w}_n(t) \equiv w(2t) + w(2t + \frac{1}{n})$ *for all* t.

Remark B.2.20 *Observe that* $\widetilde{w}_n(t) = \widehat{w}_n(t)$ *for* $0 < t < \frac{1}{2} - \frac{1}{2n}$. $\triangleleft$

Proof:

$$\begin{aligned}
\frac{1}{n} \sum_{j=1}^{\frac{n}{2}} \frac{j}{n} \widetilde{w}_n \left(\frac{j}{n}\right) &= \frac{1}{n} \sum_{j=1}^{\frac{n}{2}-1} \frac{j}{n} \widetilde{w}_n \left(\frac{j}{n}\right) + \frac{1}{2n} \widetilde{w}_n \left(\frac{1}{2}\right)\\
&= \frac{1}{n} \sum_{j=1}^{\frac{n}{2}-1} \frac{j}{n} \widehat{w}_n \left(\frac{j}{n}\right) + \frac{w(1)}{2n}\\
&= \frac{1}{2}\, h \sum_{j=1}^{\frac{n}{2}-1} f(x_j) + \frac{w(1)}{2n}\\
&\quad \text{(where } h = 2/n,\ x_j = 2j/n,\ a = 0,\ b = 1, \text{ and } f(x) = x/2 \widehat{w}_n(x/2))\\
&= \frac{1}{2} \left[I(f) - \frac{h}{2} \big(f(a) + f(b)\big) + \mathrm{O}(1/n^2) \right] + \frac{w(1)}{2n}\\
&\quad \text{(by the Trapezoid Rule)}\\
&= \frac{1}{2} \left[\int_0^1 \frac{t}{2} \widehat{w}_n \left(\frac{t}{2}\right) dt - \frac{1}{n} \big(f(0) + f(1)\big) \right] + \mathrm{O}(1/n^2) + \frac{w(1)}{2n}\\
&= \int_0^{1/2} u \widehat{w}_n(u)\, du - \frac{1}{4n} \widehat{w}_n \left(\frac{1}{2}\right) + \frac{w(1)}{2n} + \mathrm{O}(1/n^2)\\
&= \int_0^{1/2} u \widehat{w}_n(u)\, du + \frac{w(1) - w(1 + 1/n)}{4n} + \mathrm{O}(1/n^2). \quad \square \qquad \text{(B.2.19)}
\end{aligned}$$

Lemma B.2.21 *Under Assumption A.5,*

$$\frac{1}{n^3}\sum_{j=0}^{\frac{n}{2}}\sum_{k=0}^{\frac{n}{2}}(j\vee k)\,\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{w}_n\left(\frac{k}{n}\right)=2\int_0^{1/2}u\widehat{w}_n(u)\widehat{W}_n(u)\,du-\frac{1}{2n}\widehat{w}_n\left(\frac{1}{2}\right)W$$
$$+\frac{\widehat{w}_n(0)}{n}\int_0^{1/2}u\widehat{w}_n(u)\,du+\frac{w(1)W}{n}+\mathrm{O}(1/n^2). \tag{B.2.20}$$

Proof: Let us begin by manipulating the left-hand side of Equation (B.2.20).

$$\begin{aligned}
&\frac{1}{n^3}\sum_{j=0}^{\frac{n}{2}}\sum_{k=0}^{\frac{n}{2}}(j\vee k)\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{w}_n\left(\frac{k}{n}\right)\\
&=\frac{1}{n^3}\Bigg\{\sum_{j=0}^{\frac{n}{2}-1}\sum_{k=0}^{\frac{n}{2}-1}(j\vee k)\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{w}_n\left(\frac{k}{n}\right)\\
&\quad+2\sum_{j=0}^{\frac{n}{2}-1}\left(j\vee\frac{n}{2}\right)\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{w}_n\left(\frac{1}{2}\right)+\left(\frac{n}{2}\vee\frac{n}{2}\right)\widetilde{w}_n\left(\frac{1}{2}\right)\widetilde{w}_n\left(\frac{1}{2}\right)\Bigg\}\\
&=\frac{1}{n^3}\Bigg\{\sum_{j=0}^{\frac{n}{2}-1}\sum_{k=0}^{\frac{n}{2}-1}(j\vee k)\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{w}_n\left(\frac{k}{n}\right)+nw(1)\sum_{j=0}^{\frac{n}{2}-1}\widetilde{w}_n\left(\frac{j}{n}\right)+\frac{n}{2}\widetilde{w}_n^2\left(\frac{1}{2}\right)\Bigg\}\\
&=\frac{1}{n^3}\sum_{j=0}^{\frac{n}{2}-1}\sum_{k=0}^{\frac{n}{2}-1}(j\vee k)\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{w}_n\left(\frac{k}{n}\right)+\frac{w(1)}{n^2}\sum_{j=0}^{\frac{n}{2}-1}\widetilde{w}_n\left(\frac{j}{n}\right)+\mathrm{O}(1/n^2).
\end{aligned}\tag{B.2.21}$$

Meanwhile, by definition,

$$\begin{aligned}
\widetilde{W}_{D,n}\left(\frac{1}{2}\right)&=\frac{1}{n}\sum_{j=0}^{\frac{n}{2}}\widetilde{w}_n\left(\frac{j}{n}\right)\\
&=\frac{1}{n}\sum_{j=0}^{\frac{n}{2}-1}\widetilde{w}_n\left(\frac{j}{n}\right)+\frac{1}{n}\widetilde{w}_n\left(\frac{1}{2}\right)\\
&=\frac{1}{n}\sum_{j=0}^{\frac{n}{2}-1}\widetilde{w}_n\left(\frac{j}{n}\right)+\frac{w(1)}{n}.
\end{aligned}\tag{B.2.22}$$

Plugging Expression (B.2.22) into Expression (B.2.21), we have

$$\begin{aligned}
&\frac{1}{n^3}\sum_{j=0}^{\frac{n}{2}}\sum_{k=0}^{\frac{n}{2}}(j\vee k)\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{w}_n\left(\frac{k}{n}\right)\\
&=\frac{1}{n^3}\sum_{j=0}^{\frac{n}{2}-1}\sum_{k=0}^{\frac{n}{2}-1}(j\vee k)\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{w}_n\left(\frac{k}{n}\right)+\frac{w(1)\widetilde{W}_{D,n}\left(\frac{1}{2}\right)}{n}+\mathrm{O}(1/n^2).
\end{aligned}\tag{B.2.23}$$

Plugging Lemma (B.2.18) into (B.2.23), we have

$$
\begin{aligned}
&\frac{1}{n^3}\sum_{j=0}^{\frac{n}{2}}\sum_{k=0}^{\frac{n}{2}}(j\vee k)\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{w}_n\left(\frac{k}{n}\right)\\
&=\frac{1}{n^3}\sum_{j=0}^{\frac{n}{2}-1}\sum_{k=0}^{\frac{n}{2}-1}(j\vee k)\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{w}_n\left(\frac{k}{n}\right)+\frac{w(1)W}{n}+\mathrm{O}(1/n^2)\\
&=\underbrace{\frac{2}{n^3}\sum_{j=1}^{\frac{n}{2}-1}\sum_{k=0}^{j}j\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{w}_n\left(\frac{k}{n}\right)}_{\text{case } k<j \text{ and symmetry}}-\underbrace{\frac{1}{n^3}\sum_{j=0}^{\frac{n}{2}-1}j\widetilde{w}_n^2\left(\frac{j}{n}\right)}_{\text{case } k=j}+\frac{w(1)W}{n}+\mathrm{O}(1/n^2)\\
&=\frac{2}{n}\sum_{j=1}^{\frac{n}{2}-1}\frac{j}{n}\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{W}_{D,n}\left(\frac{j}{n}\right)-\frac{1}{n^2}\sum_{j=0}^{\frac{n}{2}-1}\frac{j}{n}\widetilde{w}_n^2\left(\frac{j}{n}\right)+\frac{w(1)W}{n}+\mathrm{O}(1/n^2), \qquad \text{(B.2.24)}
\end{aligned}
$$

where

$$
\begin{aligned}
&\frac{1}{n^2}\sum_{j=0}^{\frac{n}{2}-1}\frac{j}{n}\widetilde{w}_n^2\left(\frac{j}{n}\right)\\
&=\frac{1}{n^2}\sum_{j=1}^{\frac{n}{2}-1}\frac{j}{n}\widehat{w}_n^2\left(\frac{j}{n}\right)\\
&=\frac{1}{2n}h\sum_{j=1}^{\frac{n}{2}-1}f(x_j)\\
&\qquad\text{(where } h=2/n,\ x_j=2j/n,\ a=0,\ b=1, \text{ and } f(x)=(x/2)\widehat{w}_n^2(x/2))\\
&=\frac{1}{2n}\left[I(f)-\frac{h}{2}\big(f(a)+f(b)\big)+\mathrm{O}(1/n^2)\right]\\
&\qquad\text{(by the Trapezoid Rule)}\\
&=\frac{1}{2n}\left[\int_0^1\frac{t}{2}\widehat{w}_n^2\left(\frac{t}{2}\right)dt-\frac{1}{n}\big(f(0)+f(1)\big)+\mathrm{O}(1/n^2)\right]\\
&=\frac{1}{2n}\left[2\int_0^{1/2}u\widehat{w}_n^2(u)\,du-\frac{1}{2n}\widehat{w}_n^2\left(\frac{1}{2}\right)+\mathrm{O}(1/n^2)\right]\\
&=\frac{1}{n}\int_0^{1/2}u\widehat{w}_n^2(u)\,du+\mathrm{O}(1/n^2). \qquad \text{(B.2.25)}
\end{aligned}
$$

Now, define for $0\le t\le 1/2$,

$$\widehat{W}_{D,n}(t)\equiv\frac{1}{n}\sum_{k=0}^{\lfloor nt\rfloor}\widehat{w}_n(\frac{k}{n}).$$

Then, for $0 \le j \le \frac{n}{2} - 1$, we have

$$\begin{aligned}
\widetilde{W}_{D,n}\left(\frac{j}{n}\right) &= \frac{1}{n}\sum_{k=0}^{j} \widetilde{w}_n\left(\frac{k}{n}\right) \\
&= \frac{1}{n}\sum_{k=0}^{j} \widehat{w}_n\left(\frac{k}{n}\right) \\
&= \widehat{W}_{D,n}\left(\frac{j}{n}\right). \qquad \text{(B.2.26)}
\end{aligned}$$

Therefore, by the Trapezoid Rule,

$$\begin{aligned}
\widehat{W}_{D,n}\left(\frac{j}{n}\right) &= \frac{1}{n}\sum_{k=0}^{j} \widehat{w}_n\left(\frac{k}{n}\right) \\
&= \int_0^{j/n} \widehat{w}_n(t)\,dt + \frac{\widehat{w}_n(0) + \widehat{w}_n\left(\frac{j}{n}\right)}{2n} + \mathrm{O}(1/n^2) \\
&= \widehat{W}_n\left(\frac{j}{n}\right) + \frac{\widehat{w}_n(0) + \widehat{w}_n\left(\frac{j}{n}\right)}{2n} + \mathrm{O}(1/n^2), \qquad \text{(B.2.27)}
\end{aligned}$$

where $\widehat{W}_n(t) \equiv \int_0^t \widehat{w}_n(s)\,ds$.

Finally, we can address the remaining term in expression (B.2.24). We have

$$\begin{aligned}
&\frac{2}{n}\sum_{j=1}^{\frac{n}{2}-1} \frac{j}{n}\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{W}_{D,n}\left(\frac{j}{n}\right) \\
&= \frac{2}{n}\sum_{j=1}^{\frac{n}{2}-1} \frac{j}{n}\widehat{w}_n\left(\frac{j}{n}\right)\widehat{W}_{D,n}\left(\frac{j}{n}\right) \quad \text{(by (B.2.26))} \\
&= \frac{2}{n}\sum_{j=1}^{\frac{n}{2}-1} \frac{j}{n}\widehat{w}_n\left(\frac{j}{n}\right)\left[\widehat{W}_n\left(\frac{j}{n}\right) + \frac{1}{2n}\left(\widehat{w}_n(0) + \widehat{w}_n\left(\frac{j}{n}\right)\right) + \mathrm{O}(1/n^2)\right] \quad \text{(by (B.2.27))} \\
&= \frac{2}{n}\sum_{j=1}^{\frac{n}{2}-1} \frac{j}{n}\widehat{w}_n\left(\frac{j}{n}\right)\widehat{W}_n\left(\frac{j}{n}\right) + \frac{\widehat{w}_n(0)}{n^2}\sum_{j=1}^{\frac{n}{2}-1} \frac{j}{n}\widehat{w}_n\left(\frac{j}{n}\right) \\
&\quad + \frac{1}{n^2}\sum_{j=1}^{\frac{n}{2}-1} \frac{j}{n}\widehat{w}_n^2\left(\frac{j}{n}\right) + \mathrm{O}(1/n^2) \\
&= \frac{2}{n}\sum_{j=1}^{\frac{n}{2}-1} \frac{j}{n}\widehat{w}_n\left(\frac{j}{n}\right)\widehat{W}_n\left(\frac{j}{n}\right) + \frac{\widehat{w}_n(0)}{n}\left[\frac{1}{n}\sum_{j=1}^{\frac{n}{2}} \frac{j}{n}\widehat{w}_n\left(\frac{j}{n}\right) - \frac{\widehat{w}_n\left(\frac{1}{2}\right)}{2n}\right] \\
&\quad + \frac{1}{n^2}\sum_{j=1}^{\frac{n}{2}-1} \frac{j}{n}\widehat{w}_n^2\left(\frac{j}{n}\right) + \mathrm{O}(1/n^2) \\
&= \frac{2}{n}\sum_{j=1}^{\frac{n}{2}-1} \frac{j}{n}\widehat{w}_n\left(\frac{j}{n}\right)\widehat{W}_n\left(\frac{j}{n}\right) + \frac{\widehat{w}_n(0)}{n}\int_0^{1/2} u\widehat{w}_n(u)\,du
\end{aligned}$$

$$+\frac{1}{n^2}\sum_{j=1}^{\frac{n}{2}-1}\frac{j}{n}\widehat{w}_n^2\left(\frac{j}{n}\right)+\mathrm{O}(1/n^2)\quad \text{(by Equation (B.2.19))}$$

$$=\frac{2}{n}\sum_{j=1}^{\frac{n}{2}-1}\frac{j}{n}\widehat{w}_n\left(\frac{j}{n}\right)\widehat{W}_n\left(\frac{j}{n}\right)+\frac{\widehat{w}_n(0)}{n}\int_0^{1/2}u\widehat{w}_n(u)\,du+\frac{1}{n}\int_0^{1/2}u\widehat{w}_n^2(u)\,du+\mathrm{O}(1/n^2), \tag{B.2.28}$$

by (B.2.25), where

$$\frac{2}{n}\sum_{j=1}^{\frac{n}{2}-1}\frac{j}{n}\widehat{w}_n\left(\frac{j}{n}\right)\widehat{W}_n\left(\frac{j}{n}\right)$$

$$=h\sum_{j=1}^{\frac{n}{2}-1}f(x_j)$$

(where $h=2/n$, $x_j=2j/n$, $a=0$, $b=1$, and $f(x)=(x/2)\widehat{w}_n\left(x/2\right)\widehat{W}_n\left(x/2\right)$)

$$=I(f)-\frac{h}{2}\big(f(a)+f(b)\big)+O(1/n^2)$$

(by the Trapezoid Rule)

$$=\int_0^1\frac{t}{2}\widehat{w}_n\left(\frac{t}{2}\right)\widehat{W}_n\left(\frac{t}{2}\right)\,dt-\frac{1}{n}\big(f(0)+f(1)\big)+\mathrm{O}(1/n^2)$$

$$=2\int_0^{1/2}u\widehat{w}_n(u)\widehat{W}_n(u)\,du-\frac{1}{2n}\widehat{w}_n\left(\frac{1}{2}\right)\widehat{W}_n\left(\frac{1}{2}\right)+\mathrm{O}(1/n^2). \tag{B.2.29}$$

Plugging Equation (B.2.29) into Equation (B.2.28), we have

$$\begin{aligned}&\frac{2}{n}\sum_{j=1}^{\frac{n}{2}-1}\frac{j}{n}\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{W}_{D,n}\left(\frac{j}{n}\right)\\&=2\int_0^{1/2}u\widehat{w}_n(u)\widehat{W}_n(u)\,du-\frac{1}{2n}\widehat{w}_n\left(\frac{1}{2}\right)\widehat{W}_n\left(\frac{1}{2}\right)\\&\qquad+\frac{\widehat{w}_n(0)}{n}\int_0^{1/2}u\widehat{w}_n(u)\,du+\frac{1}{n}\int_0^{1/2}u\widehat{w}_n^2(u)\,du+\mathrm{O}(1/n^2),\end{aligned} \tag{B.2.30}$$

and plugging Equation (B.2.30) into Equation (B.2.24) we have

$$\begin{aligned}&\frac{1}{n^3}\sum_{j=0}^{\frac{n}{2}}\sum_{k=0}^{\frac{n}{2}}(j\vee k)\widetilde{w}_n\left(\frac{j}{n}\right)\widetilde{w}_n\left(\frac{k}{n}\right)\\&=2\int_0^{1/2}u\widehat{w}_n(u)\widehat{W}_n(u)\,du-\frac{1}{2n}\widehat{w}_n\left(\frac{1}{2}\right)\widehat{W}_n\left(\frac{1}{2}\right)\\&\quad+\frac{\widehat{w}_n(0)}{n}\int_0^{1/2}u\widehat{w}_n(u)\,du+\frac{1}{n}\int_0^{1/2}u\widehat{w}_n^2(u)\,du\\&\quad-\frac{1}{n}\int_0^{1/2}u\widehat{w}_n^2(u)\,du+\frac{w(1)W}{n}+\mathrm{O}(1/n^2)\end{aligned}$$

$$
\begin{aligned}
&= 2\int_0^{1/2} u\widehat{w}_n(u)\widehat{W}_n(u)\,du - \frac{1}{2n}\widehat{w}_n\left(\frac{1}{2}\right)\widehat{W}_n\left(\frac{1}{2}\right) \\
&\quad + \frac{\widehat{w}_n(0)}{n}\int_0^{1/2} u\widehat{w}_n(u)\,du + \frac{w(1)W}{n} + \mathrm{O}(1/n^2) \\
&= 2\int_0^{1/2} u\widehat{w}_n(u)\widehat{W}_n(u)\,du - \frac{1}{2n}\widehat{w}_n\left(\frac{1}{2}\right)W \\
&\quad + \frac{\widehat{w}_n(0)}{n}\int_0^{1/2} u\widehat{w}_n(u)\,du + \frac{w(1)W}{n} + \mathrm{O}(1/n^2) \qquad \text{(B.2.31)}
\end{aligned}
$$

by Lemma (B.2.18). □

Lemma B.2.22 *Under Assumption A.5,*

$$
\begin{aligned}
\frac{1}{n^3}\sum_{j=1}^{\frac{n}{2}}\sum_{k=1}^{\frac{n}{2}}(j\vee k)\,\breve{w}_n\left(\frac{j}{n}\right)\breve{w}_n\left(\frac{k}{n}\right) &= 2\int_0^{\frac{1}{2}} u\breve{w}_n(u)\breve{W}_n(u)\,du + \frac{1}{2n}\breve{w}_n\left(\frac{1}{2}\right)W \\
&\quad - \frac{\breve{w}_n(0)}{n}\int_0^{\frac{1}{2}} u\breve{w}_n(u)\,du + \mathrm{O}(1/n^2).
\end{aligned}
$$

Proof: Let us start by manipulating the following sum

$$
\begin{aligned}
&\frac{1}{n^3}\sum_{j=1}^{\frac{n}{2}}\sum_{k=1}^{\frac{n}{2}}(j\vee k)\,\breve{w}_n\left(\frac{j}{n}\right)\breve{w}_n\left(\frac{k}{n}\right) \\
&= \frac{2}{n^3}\underbrace{\sum_{j=2}^{\frac{n}{2}}\sum_{k=1}^{j}(j\vee k)\,\breve{w}_n\left(\frac{j}{n}\right)\breve{w}_n\left(\frac{k}{n}\right)}_{\text{case } k\le j} - \frac{1}{n^3}\underbrace{\sum_{j=1}^{\frac{n}{2}} j\breve{w}_n^2\left(\frac{j}{n}\right)}_{\text{case } k=j} \\
&= \frac{2}{n}\sum_{j=2}^{\frac{n}{2}}\frac{j}{n}\breve{w}_n\left(\frac{j}{n}\right)\frac{1}{n}\sum_{k=1}^{j}\breve{w}_n\left(\frac{k}{n}\right) - \frac{1}{n^3}\sum_{j=1}^{\frac{n}{2}} j\breve{w}_n^2\left(\frac{j}{n}\right) \\
&= \frac{2}{n}\sum_{j=2}^{\frac{n}{2}}\frac{j}{n}\breve{w}_n\left(\frac{j}{n}\right)\breve{W}_{D,n}\left(\frac{j}{n}\right) - \frac{1}{n^3}\sum_{j=1}^{\frac{n}{2}} j\breve{w}_n^2\left(\frac{j}{n}\right) \qquad \text{(B.2.32)}
\end{aligned}
$$

where

$$
\breve{W}_{D,n}(t) \equiv \frac{1}{n}\sum_{k=1}^{\lfloor nt\rfloor}\breve{w}_n\left(\frac{k}{n}\right) \quad \text{for } 1/n \le t \le 1/2,
$$

and $\breve{W}_{D,n}(0) \equiv 0$. Then, for $1 \le j \le \frac{n}{2}$, we have

$$
\breve{W}_{D,n}\left(\frac{j}{n}\right) = \frac{1}{n}\sum_{k=1}^{j}\breve{w}_n\left(\frac{k}{n}\right) = \frac{1}{n}\sum_{k=1}^{j}\breve{w}_n\left(\frac{k}{n}\right) = \breve{W}_{D,n}\left(\frac{j}{n}\right), \qquad \text{(B.2.33)}
$$

where $\check{w}_n(t) \equiv w(2t) + w(2t - 1/n)$ for all t, and $\check{W}_{D,n}(j/n) \equiv 1/n \sum_{k=1}^{j} \check{w}_n(k/n)$. Observe that $\check{w}_n(t) = \check{w}_n(t)$ for all $\frac{1}{2n} < t \leq \frac{1}{2}$. Therefore, by the Trapezoid Rule,

$$\begin{aligned}
\check{W}_{D,n}\left(\frac{j}{n}\right) &= \frac{1}{n}\sum_{k=1}^{j} \check{w}_n\left(\frac{k}{n}\right) \\
&= \frac{1}{n}\sum_{k=0}^{j} \check{w}_n\left(\frac{k}{n}\right) - \frac{\check{w}_n(0)}{n} \\
&= \int_0^{j/n} \check{w}_n(t)\,dt + \frac{\check{w}_n(0) + \check{w}_n\left(\frac{j}{n}\right)}{2n} - \frac{\check{w}_n(0)}{n} + \mathrm{O}(1/n^2) \\
&= \check{W}_n\left(\frac{j}{n}\right) + \frac{\check{w}_n\left(\frac{j}{n}\right) - \check{w}_n(0)}{2n} + \mathrm{O}(1/n^2), \qquad \text{(B.2.34)}
\end{aligned}$$

where $\check{W}_n(t) \equiv \int_0^t \check{w}_n(s)\,ds$. Next, we analyze the first term on the right-hand side of Equation (B.2.32):

$$\begin{aligned}
\frac{2}{n}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\check{w}_n\left(\frac{j}{n}\right)\check{W}_{D,n}\left(\frac{j}{n}\right) &= \frac{2}{n}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\check{w}_n\left(\frac{j}{n}\right)\check{W}_{D,n}\left(\frac{j}{n}\right) \\
&\quad\text{(by Equation (B.2.33))} \\
&= \frac{2}{n}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\check{w}_n\left(\frac{j}{n}\right)\left[\check{W}_n\left(\frac{j}{n}\right) + \frac{\check{w}_n\left(\frac{j}{n}\right) - \check{w}_n(0)}{2n} + \mathrm{O}(1/n^2)\right] \\
&\quad\text{(by Equation (B.2.34))} \\
&= \frac{2}{n}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\check{w}_n\left(\frac{j}{n}\right)\check{W}_n\left(\frac{j}{n}\right) + \frac{1}{n^2}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\check{w}_n^2\left(\frac{j}{n}\right) \\
&\quad - \frac{\check{w}_n(0)}{n^2}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\check{w}_n\left(\frac{j}{n}\right) + \mathrm{O}(1/n^2). \qquad \text{(B.2.35)}
\end{aligned}$$

Therefore, plugging Equation (B.2.35) into Equation (B.2.32) we have

$$\begin{aligned}
&\frac{1}{n^3}\sum_{j=1}^{\frac{n}{2}}\sum_{k=1}^{\frac{n}{2}} (j \vee k)\,\check{w}_n\left(\frac{j}{n}\right)\check{w}_n\left(\frac{k}{n}\right) \\
&= \frac{2}{n}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\check{w}_n\left(\frac{j}{n}\right)\check{W}_{D,n}\left(\frac{j}{n}\right) - \frac{1}{n^3}\sum_{j=1}^{\frac{n}{2}} j\check{w}_n^2\left(\frac{j}{n}\right) \\
&= \frac{2}{n}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\check{w}_n\left(\frac{j}{n}\right)\check{W}_n\left(\frac{j}{n}\right) + \frac{1}{n^2}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\check{w}_n^2\left(\frac{j}{n}\right) - \frac{\check{w}_n(0)}{n^2}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\check{w}_n\left(\frac{j}{n}\right) + \mathrm{O}(1/n^2) \\
&\quad - \frac{1}{n^3}\sum_{j=1}^{\frac{n}{2}} j\check{w}_n^2\left(\frac{j}{n}\right)
\end{aligned}$$

$$= \frac{2}{n}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\breve{w}_n\left(\frac{j}{n}\right)\breve{W}_n\left(\frac{j}{n}\right) - \frac{\breve{w}_n(0)}{n^2}\sum_{j=1}^{\frac{n}{2}} \frac{j}{n}\breve{w}_n\left(\frac{j}{n}\right) + \mathrm{O}(1/n^2) \tag{B.2.36}$$

where

$$\begin{aligned}
&\frac{2}{n}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\breve{w}_n\left(\frac{j}{n}\right)\breve{W}_n\left(\frac{j}{n}\right)\\
&= \frac{2}{n}\sum_{j=2}^{\frac{n}{2}-1} \frac{j}{n}\breve{w}_n\left(\frac{j}{n}\right)\breve{W}_n\left(\frac{j}{n}\right) + \frac{1}{n}\breve{w}_n\left(\frac{1}{2}\right)\breve{W}_n\left(\frac{1}{2}\right)\\
&= h\sum_{j=1}^{\frac{n}{2}-1} f(x_j) + \frac{1}{n}\breve{w}_n\left(\frac{1}{2}\right)\breve{W}_n\left(\frac{1}{2}\right)\\
&\qquad \text{(for } h = 2/n,\ x_j = 2j/n,\ a = 0,\ b = 1, \text{ and}\\
&\qquad f(x) = (x/2)\breve{w}_n(x/2)\breve{W}_n(x/2))\\
&= I(f) - \frac{h}{2}(f(0) + f(1)) + \mathrm{O}(1/n^2) + \frac{1}{n}\breve{w}_n\left(\frac{1}{2}\right)\breve{W}_n\left(\frac{1}{2}\right)\\
&= \int_0^1 \frac{t}{2}\breve{w}_n\left(\frac{t}{2}\right)\breve{W}_n\left(\frac{t}{2}\right)\,dt - \frac{1}{2n}\breve{w}_n\left(\frac{1}{2}\right)\breve{W}_n\left(\frac{1}{2}\right)\\
&\quad + \frac{1}{n}\breve{w}_n\left(\frac{1}{2}\right)\breve{W}_n\left(\frac{1}{2}\right) + \mathrm{O}(1/n^2)\\
&= 2\int_0^{\frac{1}{2}} u\breve{w}_n(u)\breve{W}_n(u)\,du + \frac{1}{2n}\breve{w}_n\left(\frac{1}{2}\right)\breve{W}_n\left(\frac{1}{2}\right) + \mathrm{O}(1/n^2),
\end{aligned} \tag{B.2.37}$$

and

$$\begin{aligned}
\breve{W}_n(u) &= \int_0^u \breve{w}_n(s)\,ds\\
&= \int_0^u \left(w(2s) + w\left(2s - \frac{1}{n}\right)\right)\,ds\\
&= \frac{1}{2}\int_0^{2u} w(t)\,dt + \frac{1}{2}\int_{-\frac{1}{n}}^{2u-\frac{1}{n}} w(t)\,dt\\
&= \frac{1}{2}\left(\int_0^{2u} w(t)\,dt + \int_{-\frac{1}{n}}^{0} w(t)\,dt + \int_0^{2u} w(t)\,dt - \int_{2u-\frac{1}{n}}^{2u} w(t)\,dt\right)\\
&= W(2u) + \mathrm{O}(1/n).
\end{aligned}$$

Evaluating $\breve{W}_n(u)$ at $u = 1/2$ we get

$$\breve{W}_n\left(\frac{1}{2}\right) = W(1) + \mathrm{O}(1/n) = W + \mathrm{O}(1/n) \tag{B.2.38}$$

and substituting Equation (B.2.38) into Equation (B.2.37) we get

$$
\begin{aligned}
&\frac{2}{n}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\breve{w}_n\left(\frac{j}{n}\right)\check{W}_n\left(\frac{j}{n}\right) \\
&= 2\int_0^{\frac{1}{2}} u\breve{w}_n(u)\check{W}_n(u)\,du + \frac{1}{2n}\breve{w}_n\left(\frac{1}{2}\right)(W + \mathrm{O}(1/n)) + \mathrm{O}(1/n^2) \\
&= 2\int_0^{\frac{1}{2}} u\breve{w}_n(u)\check{W}_n(u)\,du + \frac{1}{2n}\breve{w}_n\left(\frac{1}{2}\right)W + \mathrm{O}(1/n^2).
\end{aligned} \tag{B.2.39}
$$

Now consider

$$
\begin{aligned}
\frac{1}{n}\sum_{j=1}^{\frac{n}{2}} \frac{j}{n}\breve{w}_n\left(\frac{j}{n}\right) &= \frac{1}{n}\sum_{j=1}^{\frac{n}{2}-1} \frac{j}{n}\breve{w}_n\left(\frac{j}{n}\right) + \frac{1}{2n}\breve{w}_n\left(\frac{1}{2}\right) \\
&= \frac{h}{2}\sum_{j=1}^{\frac{n}{2}-1} f\left(x_j\right) + \frac{1}{2n}\breve{w}_n\left(\frac{1}{2}\right) \\
&\qquad \text{(taking } h = 2/n,\ x_j = 2j/n,\ a = 0,\ b = 1, \text{ and } f(x) = (x/2)\breve{w}_n(x/2)\text{)} \\
&= \frac{1}{2}\left[I(f) - \frac{h}{2}\left(f(a) + f(b)\right) + \mathrm{O}(1/n^2)\right] + \frac{1}{2n}\breve{w}_n\left(\frac{1}{2}\right) \\
&= \frac{1}{2}\left[\int_0^1 \frac{t}{2}\breve{w}_n\left(\frac{t}{2}\right)\,dt - \frac{1}{2n}\breve{w}_n\left(\frac{1}{2}\right)\right] + \frac{1}{2n}\breve{w}_n\left(\frac{1}{2}\right) + \mathrm{O}(1/n^2) \\
&= \int_0^{\frac{1}{2}} u\breve{w}_n(u)\,du + \frac{1}{4n}\breve{w}_n\left(\frac{1}{2}\right) + \mathrm{O}(1/n^2).
\end{aligned} \tag{B.2.40}
$$

Finally, substitution of Equations (B.2.39) and (B.2.40) into Equation (B.2.36) yields

$$
\begin{aligned}
\frac{1}{n^3}\sum_{j=1}^{\frac{n}{2}}\sum_{k=1}^{\frac{n}{2}} (j \vee k)\,\breve{w}_n\left(\frac{j}{n}\right)\breve{w}_n\left(\frac{k}{n}\right) &= \left[\frac{2}{n}\sum_{j=2}^{\frac{n}{2}} \frac{j}{n}\breve{w}_n\left(\frac{j}{n}\right)\check{W}_n\left(\frac{j}{n}\right)\right] \\
&\quad - \frac{\breve{w}_n(0)}{n}\left[\frac{1}{n}\sum_{j=1}^{\frac{n}{2}} \frac{j}{n}\breve{w}_n\left(\frac{j}{n}\right)\right] + \mathrm{O}(1/n^2) \\
&= \left[2\int_0^{\frac{1}{2}} u\breve{w}_n(u)\check{W}_n(u)\,du + \frac{1}{2n}\breve{w}_n\left(\frac{1}{2}\right)W + \mathrm{O}(1/n^2)\right] \\
&\quad - \frac{\breve{w}_n(0)}{n}\left[\int_0^{\frac{1}{2}} u\breve{w}_n(u)\,du + \frac{1}{4n}\breve{w}_n\left(\frac{1}{2}\right) + \mathrm{O}(1/n^2)\right] \\
&= 2\int_0^{\frac{1}{2}} u\breve{w}_n(u)\check{W}_n(u)\,du + \frac{1}{2n}\breve{w}_n\left(\frac{1}{2}\right)W \\
&\quad - \frac{\breve{w}_n(0)}{n}\int_0^{\frac{1}{2}} u\breve{w}_n(u)\,du + \mathrm{O}(1/n^2). \quad \square
\end{aligned}
$$

B.3 Proof of Theorem 3.2.3

We start by noticing that computing the expected value of $A_1(w,n) = N_1^2(w,n)$ is equivalent to computing the variance of $N_1(w,n)$, since $N_1(w,n)$ has zero mean. Therefore,

$$\begin{aligned}
\mathrm{E}(N_1^2(w,n)) &= \mathrm{Cov}\left[N_1(w,n), N_1(w,n)\right] \\
&= \mathrm{Cov}\left[\frac{1}{n}\sum_{j=1}^{n}\sigma w\left(\frac{j}{n}\right)T_1\left(\frac{j}{n}\right), \frac{1}{n}\sum_{k=1}^{n}\sigma w\left(\frac{k}{n}\right)T_1\left(\frac{k}{n}\right)\right] \\
&= \frac{\sigma^2}{n^2}\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)\mathrm{Cov}\left(T_1\left(\frac{j}{n}\right), T_1\left(\frac{k}{n}\right)\right).
\end{aligned}$$

Replacing $T_1(\cdot)$ by its expression in Lemma 2.3.2, applying the distributive properties of the covariance function, and gathering similar terms, we get

$$\begin{aligned}
\mathrm{E}(N_1^2(w,n)) &= \frac{1}{n^3}\left[\sum_{k=1}^{n}\left[\frac{k}{n}-1\right] w\left(\frac{k}{n}\right)\right]^2 V(n) \\
&- \frac{2}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{j}{n}-1\right] w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)\mathrm{Cov}\left[Z_n, Z_{\lfloor \frac{k}{2}\rfloor}\right] \\
&+ \frac{2}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{j}{n}-1\right] w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)\mathrm{Cov}\left[Z_n, Z_{\lfloor n-\frac{k}{2}\rfloor}\right] \\
&- \frac{2}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)\mathrm{Cov}\left[Z_{\lfloor \frac{j}{2}\rfloor}, Z_{\lfloor n-\frac{k}{2}\rfloor}\right] \\
&+ \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)\mathrm{Cov}\left[Z_{\lfloor \frac{j}{2}\rfloor}, Z_{\lfloor \frac{k}{2}\rfloor}\right] \\
&+ \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)\mathrm{Cov}\left[Z_{\lfloor n-\frac{j}{2}\rfloor}, Z_{\lfloor n-\frac{k}{2}\rfloor}\right].
\end{aligned}$$

Next, since $n \geq \left\lfloor \frac{k}{2}\right\rfloor$, $n \geq \left\lfloor n - \frac{k}{2}\right\rfloor$, and $\left\lfloor \frac{j}{2}\right\rfloor \leq \left\lfloor n-\frac{k}{2}\right\rfloor$, for $k = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, n$, we apply Equation (B.1.4) to all covariance terms, and Equation (B.1.1) to $V(n)$ to get

$$\begin{aligned}
&\mathrm{E}(N_1^2(w,n)) \\
&= \frac{1}{n^3}\left[\sum_{k=1}^{n}\left[\frac{k}{n}-1\right] w\left(\frac{k}{n}\right)\right]^2 \left[n\sigma^2 + \gamma + 2R(\infty, n)\right] \\
&- \frac{2}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{j}{n}-1\right] w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right)\left(\left\lfloor\frac{k}{2}\right\rfloor\sigma^2 + \frac{\gamma}{2} + R\left(n, \left\lfloor\frac{k}{2}\right\rfloor\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left[\frac{j}{n}-1\right]w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)\left(\left\lfloor n-\frac{k}{2}\right\rfloor\sigma^2+\frac{\gamma}{2}+R\left(n,\left\lfloor n-\frac{k}{2}\right\rfloor\right)\right)\\
&-\frac{2}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)\left(\left\lfloor\frac{j}{2}\right\rfloor\sigma^2+\frac{\gamma}{2}+R\left(\left\lfloor n-\frac{k}{2}\right\rfloor,\left\lfloor\frac{j}{2}\right\rfloor\right)\right)\\
&+\frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)\left(\left\{\left\lfloor\frac{j}{2}\right\rfloor\wedge\left\lfloor\frac{k}{2}\right\rfloor\right\}\sigma^2+\frac{\gamma}{2}\right)\\
&+\frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)R\left(\left\{\left\lfloor\frac{j}{2}\right\rfloor\vee\left\lfloor\frac{k}{2}\right\rfloor\right\},\left\{\left\lfloor\frac{j}{2}\right\rfloor\wedge\left\lfloor\frac{k}{2}\right\rfloor\right\}\right)\\
&+\frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)\left(\left\{\left\lfloor n-\frac{j}{2}\right\rfloor\wedge\left\lfloor n-\frac{k}{2}\right\rfloor\right\}\sigma^2+\frac{\gamma}{2}\right)\\
&+\frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)R\left(\left\{\left\lfloor n-\frac{j}{2}\right\rfloor\vee\left\lfloor n-\frac{k}{2}\right\rfloor\right\},\left\{\left\lfloor n-\frac{j}{2}\right\rfloor\wedge\left\lfloor n-\frac{k}{2}\right\rfloor\right\}\right).
\end{aligned}
$$

Let now us compute the coefficient of σ^2, the coefficient of γ, and the remaining terms separately. First, the coefficient of σ^2 equals

$$
\begin{aligned}
&\frac{1}{n^2}\left[\sum_{k=1}^{n}\left(\frac{k-n}{n}\right)w\left(\frac{k}{n}\right)\right]^2\\
&\quad-\frac{2}{n^2}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j-n}{n}\right)w\left(\frac{j}{n}\right)\right]\left[\sum_{k=1}^{n}\left\lfloor\frac{k}{2}\right\rfloor w\left(\frac{k}{n}\right)\right]\\
&\quad+\frac{2}{n^2}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j-n}{n}\right)w\left(\frac{j}{n}\right)\right]\left[\sum_{k=1}^{n}\left\lfloor n-\frac{k}{2}\right\rfloor w\left(\frac{k}{n}\right)\right]\\
&\quad-\frac{2}{n^2}\left[\sum_{j=1}^{n}\left\lfloor\frac{j}{2}\right\rfloor w\left(\frac{j}{n}\right)\right]\left[\frac{1}{n}\sum_{k=1}^{n}w\left(\frac{k}{n}\right)\right]\\
&\quad+\frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor\frac{j}{2}\right\rfloor\wedge\left\lfloor\frac{k}{2}\right\rfloor\right)w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right)\\
&\quad+\frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor n-\frac{j}{2}\right\rfloor\wedge\left\lfloor n-\frac{k}{2}\right\rfloor\right)w\left(\frac{j}{n}\right)w\left(\frac{k}{n}\right).
\end{aligned}
$$

After some algebra the coefficient of σ^2 becomes

$$
\begin{aligned}
&\left[\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k-n}{n}\right)w\left(\frac{k}{n}\right)\right]^2\\
&\quad+\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j-n}{n}\right)w\left(\frac{j}{n}\right)\right]\left[\frac{1}{n}\sum_{k=1}^{n}\frac{1}{n}\left(\left\lfloor n-\frac{k}{2}\right\rfloor-\left\lfloor\frac{k}{2}\right\rfloor\right)w\left(\frac{k}{n}\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{2}{n^2}\left[\frac{1}{n}\sum_{k=1}^{n} w\left(\frac{k}{n}\right)\right]\left[\sum_{j=1}^{n}\left\lfloor\frac{j}{2}\right\rfloor w\left(\frac{j}{n}\right)\right] \\
&+ \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor\frac{j}{2}\right\rfloor \wedge \left\lfloor\frac{k}{2}\right\rfloor\right) w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) \\
&+ \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor n-\frac{j}{2}\right\rfloor \wedge \left\lfloor n-\frac{k}{2}\right\rfloor\right) w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right).
\end{aligned}
$$

Since $\lfloor n-k/2\rfloor - \lfloor k/2\rfloor = n-k$, for $k = 1, 2, \ldots, n$, the last expression can be written as

$$
\begin{aligned}
&- \left[\frac{1}{n}\sum_{k=1}^{n}\left(\frac{j-n}{n}\right) w\left(\frac{k}{n}\right)\right]^2 \\
&- \frac{2}{n^2}\left[\frac{1}{n}\sum_{k=1}^{n} w\left(\frac{k}{n}\right)\right]\left[\sum_{j=1}^{n}\left\lfloor\frac{j}{2}\right\rfloor w\left(\frac{j}{n}\right)\right] \\
&+ \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor\frac{j}{2}\right\rfloor \wedge \left\lfloor\frac{k}{2}\right\rfloor\right) w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) \\
&+ \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n}\left(\left\lfloor n-\frac{j}{2}\right\rfloor \wedge \left\lfloor n-\frac{k}{2}\right\rfloor\right) w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right).
\end{aligned}
$$

Finally, we get the coefficient of σ^2 in $\mathrm{E}(N_1^2(w,n))$. We will call that exact coefficient $W_{D,n}^*$.

$$
\begin{aligned}
W_{D,n}^* \equiv W_{D,n}^2 - \overline{W}_{D,n}^2 &+ \frac{2w(0)}{n^3}\sum_{j=1}^{\frac{n}{2}} j\widetilde{w}_n\left(\frac{j}{n}\right) \\
&- \frac{1}{n^3}\sum_{j=1}^{\frac{n}{2}}\sum_{k=1}^{\frac{n}{2}}(j\vee k)\,\breve{w}_n\left(\frac{j}{n}\right)\breve{w}_n\left(\frac{k}{n}\right) \\
&- \frac{1}{n^3}\sum_{j=0}^{\frac{n}{2}}\sum_{k=0}^{\frac{n}{2}}(j\vee k)\,\widetilde{w}\left(\frac{j}{n}\right)\widetilde{w}\left(\frac{k}{n}\right)
\end{aligned}
\tag{B.3.1}
$$

By Lemmas B.2.1, B.2.3, B.2.4, B.2.6, B.2.16, B.2.17, B.2.19, B.2.21, and B.2.22 then we have

$$
W_{D,n}^* = W^2 + \frac{w(1)-w(0)}{n}W - \overline{W}^2 + \frac{2w(0)\overline{W}}{n} + \mathrm{O}(1/n^2)
$$

(by Lemmas B.2.16 and B.2.17)

$$
+ \frac{2w(0)}{n}\left[\int_0^{1/2} u\hat{w}_n(u)\,du + \frac{w(1)-w(1+1/n)}{4n}\right]
$$

(by Lemma B.2.19)

$$
- 2\int_0^{1/2} u\breve{w}_n(u)\breve{W}_n(u)\,du - \frac{1}{2n}\breve{w}_n\left(\frac{1}{2}\right)W + \frac{\breve{w}_n(0)}{n}\int_0^{1/2} u\breve{w}_n(u)\,du
$$

(by Lemma B.2.22)

$$- 2\int_0^{1/2} u\widehat{w}_n(u)\widehat{W}_n(u)\,du + \frac{1}{2n}\widehat{w}_n\left(\frac{1}{2}\right)W - \frac{\widehat{w}_n(0)}{n}\int_0^{1/2} u\widehat{w}_n(u)\,du - \frac{w(1)W}{n}$$

(by Lemma B.2.21)

$$= W^2 - \overline{W}^2 + \frac{w(0)}{n}(2\overline{W} - W) + \frac{W}{2n}\left(w\left(1+\frac{1}{n}\right) - w\left(1-\frac{1}{n}\right)\right)$$

$$- 2\int_0^{1/2} u\check{w}_n(u)\check{W}_n(u)\,du - 2\int_0^{1/2} u\widehat{w}_n(u)\widehat{W}_n(u)\,du$$

$$+ \frac{w(0) - w(1/n)}{n}\int_0^{1/2} u\widehat{w}_n(u)\,du + \frac{\check{w}_n(0)}{n}\int_0^{1/2} u\check{w}_n(u)\,du + \mathrm{O}(1/n^2)$$

$$= W^2 - \overline{W}^2 + \frac{w(0)}{n}(2\overline{W} - W)$$

$$- 2\int_0^{1/2} u\check{w}_n(u)\check{W}_n(u)\,du - 2\int_0^{1/2} u\widehat{w}_n(u)\widehat{W}_n(u)\,du$$

$$+ \frac{\check{w}_n(0)}{n}\int_0^{1/2} u\check{w}_n(u)\,du + \mathrm{O}(1/n^2)$$

(by the Mean Value Theorem)

$$= W_n^*, \tag{B.3.2}$$

as defined in Theorem 3.2.3.

On the other hand, the coefficient of γ in $\mathrm{E}(N_1^2(w,n))$ is

$$\frac{1}{n}\left[\frac{1}{n}\sum_{k=1}^{n}\left(\frac{j-n}{n}\right)w\left(\frac{k}{n}\right)\right]^2$$

$$- \frac{1}{n}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j-n}{n}\right)w\left(\frac{j}{n}\right)\right]\left[\frac{1}{n}\sum_{k=1}^{n}w\left(\frac{k}{n}\right)\right]$$

$$+ \frac{1}{n}\left[\frac{1}{n}\sum_{j=1}^{n}\left(\frac{j-n}{n}\right)w\left(\frac{j}{n}\right)\right]\left[\frac{1}{n}\sum_{k=1}^{n}w\left(\frac{k}{n}\right)\right]$$

$$- \frac{1}{n}\left[\frac{1}{n}\sum_{j=1}^{n}w\left(\frac{j}{n}\right)\right]\left[\frac{1}{n}\sum_{k=1}^{n}w\left(\frac{k}{n}\right)\right]$$

$$+ \frac{1}{n}\left[\frac{1}{n}\sum_{j=1}^{n}w\left(\frac{j}{n}\right)\right]\left[\frac{1}{n}\sum_{k=1}^{n}w\left(\frac{k}{n}\right)\right]$$

$$= \frac{\overline{W}_{D,n}^2}{n},$$

(by Lemma B.2.1)

$$= \frac{\overline{W}^2}{n} - \frac{2w(0)\overline{W}}{n^2} + \mathrm{O}(1/n^2)$$

(by Lemma B.2.17)

$$= \frac{\overline{W}^2}{n} + \mathrm{O}(1/n^2) \tag{B.3.3}$$

by Assumption A.5.

Similarly, the remaining terms of $\mathrm{E}(N_1^2(w,n))$ are

$$\begin{aligned}
&\frac{2\overline{W}_{D,n}^2}{n} R(\infty,n) + \frac{2\overline{W}_{D,n}}{n^2}\sum_{k=1}^{n} w\left(\frac{k}{n}\right) R\left(n, \left\lfloor\frac{k}{2}\right\rfloor\right) \\
&\quad - \frac{2\overline{W}_{D,n}}{n^2}\sum_{k=1}^{n} w\left(\frac{k}{n}\right) R\left(n, \left\lfloor n-\frac{k}{2}\right\rfloor\right) \\
&\quad - \frac{2}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) R\left(\left\lfloor n-\frac{k}{2}\right\rfloor, \left\lfloor\frac{j}{2}\right\rfloor\right) \\
&\quad + \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) R\left(\left\{\left\lfloor\frac{j}{2}\right\rfloor \vee \left\lfloor\frac{k}{2}\right\rfloor\right\}, \left\{\left\lfloor\frac{j}{2}\right\rfloor \wedge \left\lfloor\frac{k}{2}\right\rfloor\right\}\right) \\
&\quad + \frac{1}{n^3}\sum_{j=1}^{n}\sum_{k=1}^{n} w\left(\frac{j}{n}\right) w\left(\frac{k}{n}\right) R\left(\left\{\left\lfloor n-\frac{j}{2}\right\rfloor \vee \left\lfloor n-\frac{k}{2}\right\rfloor\right\}, \left\{\left\lfloor n-\frac{j}{2}\right\rfloor \wedge \left\lfloor n-\frac{k}{2}\right\rfloor\right\}\right) \\
&= \mathrm{O}(1/n^2),
\end{aligned} \tag{B.3.4}$$

by Lemmas B.2.7 to B.2.14 and Assumption A.5. Theorem 3.2.3 follows by combining Equations (B.3.2), (B.3.3), and (B.3.4) in $\mathrm{E}(N_1^2(w,n))$, and noticing that $\mathrm{O}(1/n^2)$ can be replaced by $\mathrm{o}(1/n)$. □

B.4 Notation for Theorem 4.2.1

Before presenting the proof, let us define some notation. Since the weight function $g(\cdot)$ is assumed to be continuous and bounded on $[0,1]$ (see Assumption A.6), we denote $M' \equiv \sup_{0\le t\le 1}|g(t)| < \infty$. Additionally

$$G_D\left(\frac{m}{n}\right) \equiv \frac{1}{n}\sum_{k=1}^{m} g\left(\frac{k}{n}\right) \quad \text{for } m = 1,2,\ldots,n. \tag{B.4.1}$$

$$G_D \equiv \frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right) = G_D(1). \tag{B.4.2}$$

$$\overline{G}_D \equiv \frac{1}{n}\sum_{k=1}^{n-1} G_D\left(\frac{k}{n}\right). \tag{B.4.3}$$

B.5 Proof of Theorem 4.2.1

By Definition 2.3.3, we have

$$\begin{aligned}
&\mathrm{E}(C_1(g,n)) \\
&= \mathrm{E}\left(\frac{1}{n}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\sigma T_1\left(\frac{k}{n}\right)\right]^2\right) \\
&= \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\mathrm{E}\left(\frac{k-n}{n}Z_n - Z_{\lfloor \frac{k}{2}\rfloor} - Z_{\lfloor n-\frac{k}{2}\rfloor}\right)^2 \\
&= \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\frac{k-n}{n}\right]^2 \mathrm{E}\left(Z_n^2\right) + \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\frac{k-n}{n}\right]^2 \mathrm{E}\left(Z_{\lfloor \frac{k}{2}\rfloor}^2\right) \\
&+ \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\frac{k-n}{n}\right]^2 \mathrm{E}\left(Z_{\lfloor n-\frac{k}{2}\rfloor}^2\right) - \frac{2}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\frac{k-n}{n}\right] \mathrm{E}\left(Z_n Z_{\lfloor \frac{k}{2}\rfloor}\right) \\
&- \frac{2}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right)\left[\frac{k-n}{n}\right] \mathrm{E}\left(Z_n Z_{\lfloor n-\frac{k}{2}\rfloor}\right) + \frac{2}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right) \mathrm{E}\left(Z_{\lfloor \frac{k}{n}\rfloor} Z_{\lfloor n-\frac{k}{2}\rfloor}\right).
\end{aligned}$$

Now, let us apply Lemmas B.1.2 and B.1.5. After some algebra,

$$\begin{aligned}
&\mathrm{E}(C_1(g,n)) \\
&= \left[\overline{G}_D - \frac{1}{n}\sum_{k=1}^{n}\left[\frac{k-n}{n}\right]^2 g\left(\frac{k}{n}\right)\right]\sigma^2 + \left[\frac{G_D}{n} + \frac{1}{n^2}\sum_{k=1}^{n}\left[\frac{k-n}{n}\right]^2 w\left(\frac{k}{n}\right)\right]\gamma \\
&+ \frac{R(\infty,n)}{n^2}\sum_{k=1}^{n}\left[\frac{k-n}{n}\right]^2 g\left(\frac{k}{n}\right) + \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right) R\left(\infty, \left\lfloor\frac{k}{2}\right\rfloor\right) \\
&+ \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right) R\left(\infty, \left\lfloor n-\frac{k}{2}\right\rfloor\right) - \frac{1}{n^2}\sum_{k=1}^{n}\left[\frac{k-n}{n}\right] g\left(\frac{k}{n}\right) R\left(n, \left\lfloor\frac{k}{2}\right\rfloor\right) \\
&+ \frac{1}{n^2}\sum_{k=1}^{n}\left[\frac{k-n}{n}\right] g\left(\frac{k}{n}\right) R\left(n, \left\lfloor n-\frac{k}{2}\right\rfloor\right) - \frac{1}{n^2}\sum_{k=1}^{n} g\left(\frac{k}{n}\right) R\left(\left\lfloor n-\frac{k}{2}\right\rfloor, \left\lfloor\frac{k}{2}\right\rfloor\right).
\end{aligned}$$

Similar to the proof of Theorem 3.2.3, we would like to replace the discrete approximations to integrals with the integrals themselves. Notice that Lemmas B.2.17 and B.2.18 give us the needed expressions for G_D and $\overline{G}_D$. We apply the Trapezoid Rule for integrals to $\int_0^1 (1-t)^2 g(t)dt$ to obtain

$$\frac{1}{n}\sum_{k=1}^{n-1}\left[1-\frac{k}{n}\right]^2 g\left(\frac{k}{n}\right) = \int_0^1 (1-t)^2 g(t)\,dt - \frac{g(0)}{2n} + \mathrm{O}(1/n^2).$$

The proof follows by replacing the discrete approximations to integrals with the integral themselves, making the appropriate simplifications, bounding the last six terms of $\mathrm{E}(C_1(g,n))$ using the same techniques as in the proofs of Lemmas B.2.7 to B.2.12, and noticing that $\mathrm{O}(1/n^2)$ can be replaced by $\mathrm{o}(1/n)$. $\square$

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